PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The Reflexion of Plane Gravity Waves Travelling in Water of Variable Depth

G. F. Fitz-Gerald

Phil. Trans. R. Soc. Lond. A 1976 **284**, 49-89
doi: 10.1098/rsta.1976.0099

THE REFLEXION OF PLANE GRAVITY WAVES TRAVELLING IN WATER OF VARIABLE DEPTH

By G. F. FITZ-GERALD†
Department of Mathematics, University of Western Australia
† *Present address: R.M.I.T., Melbourne, Australia*

(*Communicated by F. Ursell, F.R.S. – Received* 18 *September* 1975)

CONTENTS

A two dimensional, irrotational, linear theory is used to investigate the reflexion of an incident surface gravity wave travelling over a region of varying depth. The existence of a unique velocity potential is proved for general bottom profiles in two limiting cases, when the wavelength is either small compared with the depth or large compared with the transition width. The associated asymptotic results justify the approximations obtained by others using formal methods. Also, the class of bottom profiles for which numerical results can be achieved is extended.

1. Introduction

This paper is concerned with the reflexion of an incident surface gravity wave that travels over a region of varying depth. A two dimensional, irrotational, linear theory is considered.

The current knowledge of this subject may be discussed under four headings: a few theorems, a few exact solutions, numerical studies and results believed to be asymptotic. The theorems apply to general bottom profiles with results related solely to the reflexion and transmission coefficients. The exact solutions and numerical studies are applicable to restricted types of bottom profiles, while the asymptotic results, for large or small wavelengths or small obstacle heights, are generally unproven.

It is a surprising feature of the problem that although the uniqueness of the reflexion and transmission coefficients has been proved (Kreisel 1949), there is no corresponding result for the uniqueness of the velocity field. It remains an open question whether there exist solutions, tending to zero at infinity, which describe modes trapped on some topographical feature of the bottom profile.

When approximations are made in regions where spectral questions are open, it is possible that the approximate system possesses eigensolutions, even if the exact system does not. To make progress with such problems, it seems necessary to seek particular formulations in which fictitious eigensolutions are suppressed.

The existence of a unique velocity potential is proved for general bottom profiles in the limiting cases when the wavelength is either small compared with the depth or large compared with the transition width. A general numerical procedure is devised for wavelengths of the order of the depth. These results are basic to determining the reflexion coefficient for all values of the parameters.

The basic equations are stated in § 2.1. In § 2.2 these equations are reformulated as a boundary value problem in a parallel strip. Then Fourier transform techniques are applied and two integro-differential equations are obtained for essentially the wave amplitude. It is shown that a one-to-one correspondence holds between the set of solutions of the original problem and the set of solutions of both integro-differential equations. However, approximations to these integro-differential equations are not necessarily equivalent. The key to obtaining the results of this paper is thus to find, for appropriate ranges of the parameters, suitable approximation forms by choosing particular linear combinations of the two integro-differential equations for which convergent iteration schemes can be developed.

In § 3.1 the existence of a unique solution of an appropriately transformed problem is proved when the wavelength is small compared with the depth by using the contraction mapping theorem. Then the existence of a unique velocity potential for a certain parameter range is established. The associated Picard iteration scheme provides asymptotic results, the leading order terms of which are essentially equivalent to those obtained in the literature using formal methods. The corresponding results for the limiting case when the wavelength is large compared with the transition width are discussed in § 3.2. A numerical iteration procedure is outlined in § 3.3 which applies when the wavelength is of the same order as the depth. The success of the process depends on selecting a suitable linear combination of the two integro-differential equations in such a way that the system of approximating algebraic equations is sufficiently well conditioned. In computing numerous results, the author encountered no serious problems in obtaining suitable linear combinations. Possible limitations of the method are discussed.

In § 4 a general discussion on the determination of the conformal transformation for a general bottom profile is given, and the conformal transformations for particular profiles of interest are discussed. The corresponding values of the reflexion coefficients are displayed graphically as functions of both the wavenumber and the ratio of transition width to wavelength.

2. The governing equations

2.1. *Basic equations and scales of motion*

Consider the incidence of a periodic surface wave train of amplitude a and angular frequency ω on a region of varying depth. Assume the velocity field is periodic with the frequency of the incident wave. Let g be the acceleration due to gravity and d be a suitably chosen typical depth. Select ω^{-1} as the unit of time, a as the unit of wave amplitude, d as the unit of length and $ga\omega^{-1}$ as the unit of velocity potential.

Choose a rectangular cartesian coordinate system in such a way that the y_1-axis points vertically upwards with $y_1 = 0$ defining the undisturbed free surface of the fluid. In terms of the above scales, the coordinates of the bottom profile are $(dx_1, -db(x_1))$. Denote time by $\omega^{-1}t$, the wave amplitude by $a\zeta$ and the velocity potential by $ga\omega^{-1}\Phi$. The well-known equations which govern wave propagation in the small amplitude potential theory of two dimensional surface gravity waves are then

$$\Phi_{x_1x_1} + \Phi_{y_1y_1} = 0 \quad \text{in} \quad -\infty < x_1 < \infty, \quad -b(x_1) < y_1 < 0, \tag{2.1.1}$$

$$\Phi_{y_1} + b'(x_1)\,\Phi_{x_1} = 0 \quad \text{on} \quad y_1 = -b(x_1) \tag{2.1.2}$$

and

$$\omega^2 dg^{-1}\Phi_{tt} + \Phi_{y_1} = 0 \quad \text{on} \quad y_1 = 0 \tag{2.1.3}$$

(see, for example, Wehausen & Laitone (1960)). The wave amplitude is given by the formula

$$\zeta = -\Phi_t|_{y_1=0}. \tag{2.1.4}$$

When the depth is uniform, Ursell (1953) has shown that these equations give a suitable description of surface gravity wave propagation providing both the parameters ad^{-1} and $a\lambda^2 d^{-3}$ are very much smaller than unity. Here the wavelength $\lambda = 2\pi\tilde{k}^{-1}$, where the wavenumber $\tilde{k}$ is the real positive root of the dispersion relation $\tilde{k}d \tanh \tilde{k}d = \omega^2 dg^{-1}$.

If a small amplitude wave travels in a fluid of variable depth, in which the depth changes are not too rapid, then it seems reasonable to apply Ursell's (1953) criterion locally. This suggests that the linear theory is applicable in those regions of varying depth where both the parameters $a\zeta(x_1, t)\,(db(x_1))^{-1}$ and $a\zeta(x_1, t)\,(\hat{\lambda}(x_1))^2\,(db(x_1))^{-3}$ are uniformly very much smaller than unity. Here the wavelength $\hat{\lambda}(x_1) = 2\pi(\tilde{k}_1(x_1))^{-1}$ where the wavenumber $\tilde{k}_1(x_1)$ is the real positive root of the local dispersion relation $\tilde{k}_1(x_1)\,d \tanh \tilde{k}_1(x_1)\,db(x_1) = \omega^2 dg^{-1}$. This form of the dispersion relation appears in Keller's (1958) formal application of the methods of geometrical optics when the bottom profile b is slowly varying. For more rapidly varying depth changes there is no *a priori* estimate of the wavelength available. There clearly must be restrictions on the depth variations to exclude the breaking of the waves. It is expected that for sufficiently small values of a, providing the other parameters are held fixed, the equations (2.1.1)–(2.1.4) will apply. An appropriate tactic would seem to be to perform the calculations, obtain the surface response and then use a typical length scale over which the surface displacement varies, in place of the wavelength $\hat{\lambda}(x_1)$, in the above condition.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Since the motion is periodic in time and the equations are linear, the time dependence is removed by expressing the function value $\Phi(x_1, y_1, t)$ in the form $\mathrm{Re}\{-\mathrm{i}\psi(x_1, y_1)\exp(\mathrm{i}(\Theta - t))\}$, where the real constant Θ is introduced to simplify the accounting for spatial phase relations in the subsequent analysis. Suppose the incident wave travels from left to right, in the direction of increasing x_1, in a fluid which approaches uniform depth as $x_1 \to \pm\infty$. For large negative values of x_1, the field consists of an incident plane wave plus a reflected plane wave, that is

$$\psi(x_1, y_1) \to \frac{\cosh \tilde{k}_-(y_1+1)}{\cosh \tilde{k}_-}(\exp(\mathrm{i}\tilde{k}_- x_1) + \tilde{r}\exp(-\mathrm{i}\tilde{k}_- x_1)) \quad (x_1 \to -\infty); \tag{2.1.5a}$$

and for large positive values of x_1, the field consists of a transmitted plane wave only; that is,

$$\psi(x_1, y_1) \to \frac{\cosh \tilde{k}_+(y_1+\epsilon)}{\cosh \tilde{k}_+\epsilon}\,\tilde{t}\exp(\mathrm{i}\tilde{k}_+ x_1) \quad (x_1 \to +\infty), \tag{2.1.5b}$$

where $d^{-1}\tilde{k}_\pm$ are the wavenumbers of waves travelling in fluid of uniform depth ϵd (corresponding to that at $x_1 = +\infty$) and d (corresponding to that at $x_1 = -\infty$). The quantities $\tilde{r}$ and $\tilde{t}$ are constants to be determined; the values of $|\tilde{r}|^2$ and $|\tilde{t}|^2$ are called the reflexion and transmission coefficients respectively.

Kreisel (1949) has shown that in the two dimensional theory of surface gravity waves travelling in water of variable depth, the reflexion coefficient is the same for waves incident from either the deep or the shallow end providing only that the approach to uniform depth at both ends is exponential. The relation between the corresponding transmission coefficients is derived in Newman (1965). Thus with no loss in generality the deeper end, if there is one, is taken at $x_1 = -\infty$ so that the parameter $\epsilon \in (0, 1]$.

It is instructive to perform an energy argument based on the divergence $\nabla\cdot(\Psi^*\nabla\Psi)$, where Ψ is the difference $\psi_1 - \psi_2$ of two possible solutions of the equations (2.1.1), (2.1.2) and (2.1.3) and Ψ^* is the complex conjugate of Ψ. The Green–Riemann formula for plane integrals and the orthogonality results of Kreisel (1949) imply that

$$\begin{aligned} 0 &= \iint_D \Psi^*\nabla^2\Psi\,\mathrm{d}x_1\,\mathrm{d}y_1 \\ &= \mathrm{i}|\tilde{t}_1 - \tilde{t}_2|^2 A + \mathrm{i}|\tilde{r}_1 - \tilde{r}B_2|^2 + \omega^2 d g^{-1}\int_{-X_0}^{X_0} |\Psi|^2_{y_1=0}\,\mathrm{d}x_1 - \iint_D |\nabla\Psi|^2\,\mathrm{d}x_1\,\mathrm{d}y_1, \end{aligned} \tag{2.1.6}$$

where $|\tilde{r}_j|^2$ and $|\tilde{t}_j|^2$ are the reflexion and transmission coefficients respectively for the solutions ψ_j with $j = 1$ or 2, the constants A and B are real and positive, the domain D is the region $-X_0 \leqslant x_1 \leqslant X_0$, $-b(x_1) \leqslant y_1 \leqslant 0$ and the positive quantity X_0 is chosen large enough so that in the regions $x_1 > X_0$ and $x_1 < -X_0$ the depth is ϵd and d respectively. Subsequently the value of X_0 will be taken as $+\infty$ and the approach of the function value $b(x_1)$ to ϵ and 1 as $x_1 \to \pm\infty$ to be exponential.

The imaginary part of equation (2.1.6) implies the uniqueness result of Kreisel (1949), namely $\tilde{t}_1 = \tilde{t}_2$ and $\tilde{r}_1 = \tilde{r}_2$. The real part of equation (2.1.6) yields the result

$$\omega^2 d g^{-1}\int_{-X_0}^{X_0} |\Psi|^2_{y_1=0}\,\mathrm{d}x_1 = \iint_D |\nabla\Psi|^2\,\mathrm{d}x_1\,\mathrm{d}y_1.$$

Thus this argument does not imply the uniqueness of the solution but merely equipartition of energy. Although no solution for a one dimensional bottom profile is known that includes trapped modes, convincing physical arguments to exclude them appear to be lacking. Furthermore there are known two dimensional bottom profiles which support trapped wave modes so that the above failure of the energy method to prove uniqueness may be significant.

Even if there are no trapped modes for the original problem, care will be necessary in any approximation scheme in order to avoid the introduction of false eigenfunctions. Thus the ensuing solution procedures will be arranged so as to lead to formulations of the problem that have a unique solution which is also one solution of the original problem. Then as the reflexion coefficient is uniquely determined, even if the total solution is not, information about the reflexion coefficient can be obtained.

2.2. *Derivation of the integro-differential equations*

The boundary value problem (2.1.1), (2.1.2) and (2.1.3), for general bottom profiles b, is not attractive for direct treatment using either analytical or numerical methods because of the radiation conditions. Let the region D': $-\infty < x_1 < \infty$, $-b(x_1) < y_1 < 0$ be the z_1-plane (where $z_1 = x_1 + iy_1$) and let the parallel strip D'': $-\infty < x < \infty$, $-\pi < y < 0$ be the z-plane (where $z = x + iy$). A conformal transformation $z_1 = Q(\beta z)$ is sought which maps D'' on to D' in such a way that the points at infinity correspond, the x-axis maps to the undisturbed free surface $y_1 = 0$ and the line $y = -\pi$ maps on to the bottom profile $y_1 = -b(x_1)$. Here the parameter $\beta \in (0, 1]$ is introduced to cope with variations in the scale length of the depth variations; the value of $\beta = 1$ corresponds to a bottom profile with a discontinuous change in depth and the limiting case of β approaching zero corresponds to the 'transition width' becoming infinite. A complete definition of β, for bottom profiles of specific interest, is given in § 4. The mapping is uniquely defined by choosing one point in each plane to correspond.

In the problems under consideration, the function value $b(x_1)$ approaches ϵ and 1 exponentially as $x_1 \to \pm\infty$. Thus it is expected that near the ends the mapping function Q has the approximate behaviour

$$Q(\beta z) \simeq \pi^{-1} z \quad \text{for} \quad x \to -\infty$$

and

$$Q(\beta z) \simeq \epsilon\pi^{-1} z \quad \text{for} \quad x \to +\infty.$$

In Kreisel (1949) this assertion is proved when the depths at $x_1 = \pm\infty$ are the same and the obstacle is finite in length, that is there exists an $\hat{X}_0 > 0$ such that for values of $|x_1| \geqslant \hat{X}_0$ the depth function b takes its limiting value. It is not believed that the existence of an $\hat{X}_0$ and the same limiting depth are essential requirements for this result to hold, but that with considerable effort similar results could be established for the bottom profiles considered here. Consequently the radiation conditions for the transformed potential are assumed to be of the same form as those for ψ but with the wavenumbers at $x = \pm\infty$ given by the values $d^{-1}K_+ = \epsilon d^{-1}\tilde{k}_+\pi^{-1}$ and $d^{-1}K_- = d^{-1}\tilde{k}_-\pi^{-1}$ respectively.

Let the transformed potential ϕ be defined by the equation

$$\phi(x, y) = \psi(x_1(\beta x, \beta y), y_1(\beta x, \beta y)).$$

Then ϕ satisfies Laplace's equation

$$\phi_{xx} + \phi_{yy} = 0 \quad \text{in} \quad -\infty < x < \infty, \quad -\pi < y < 0, \tag{2.2.1}$$

the boundary conditions

$$\phi_y = 0 \quad \text{on} \quad y = -\pi, \tag{2.2.2}$$

$$\omega^2 d g^{-1} \left.\frac{dz_1}{dz}\right|_{y=0} \phi - \phi_y = 0 \quad \text{on} \quad y = 0, \tag{2.2.3}$$

and the radiation conditions

$$\left.\begin{aligned}\phi(x,y) &\to \frac{\cosh K_-(y+\pi)}{\cosh \pi K_-}(\exp(\mathrm{i}K_- x) + \hat{R}\exp(-\mathrm{i}K_- x)) \quad (x\to -\infty),\\ \text{and}\quad \phi(x,y) &\to \frac{\cosh K_+(y+\pi)}{\cosh \pi K_+}\, \hat{T}\exp(\mathrm{i}K_+ x) \quad (x \to +\infty).\end{aligned}\right\} \tag{2.2.4}$$

The quantities $\hat{R}$ and $\hat{T}$ are complex constants to be determined which differ from the constants $\tilde{r}$ and $\tilde{t}$ only in their arguments. To find a solution of equations (2.2.1), (2.2.2) and (2.2.3), that satisfies the radiation conditions (2.2.4), will be called problem A.

The effect of the conformal transformation appears in problem A through the real quantity $\mathrm{d}z_1/\mathrm{d}z|_{y=0}$. Let h be the real-valued analytic function defined by the equation $h(\beta x) = \mathrm{d}z_1/\mathrm{d}z|_{y=0}$. The practical determination of suitable approximations to h, for given bottom profiles b, is discussed in § 4. For the present it is assumed that h is known and to facilitate the derivation of subsequent estimates that the function h has the limiting behaviour

$$h(\beta x) = h(\pm\infty) + O(\exp(-\beta|x|)) \quad \text{as} \quad x \to \pm\infty.$$

Note that the assumptions are made on the behaviour of the function h at $x = \pm\infty$ rather than the behaviour of the depth profile b at $x_1 = \pm\infty$.

Define the Fourier transform $\bar{g}$ of a function g by the equation

$$\bar{g}(\kappa) = \int_{-\infty}^{\infty} g(x)\exp(\mathrm{i}\kappa x)\,\mathrm{d}x.$$

Although the Fourier transforms of the functions ϕ and $x \mapsto h(\beta x)\,\phi(x,0)$ should be interpreted as generalized functions, it is convenient to proceed with purely formal manipulations, interpreting the functions appropriately and then making careful *a posteriori* verifications of the results. Let $\eta = \pi^{-1}d\omega^2 g^{-1}$ and $f(x) = \phi(x,0)$. Then the formal solution of problem A is the inverse Fourier transform of the function

$$(\kappa, y) \mapsto \bar{\phi}(\kappa, y) = \bar{f}(\kappa)\frac{\cosh\kappa(y+\pi)}{\cosh\pi\kappa}, \tag{2.2.5}$$

where

$$-\pi(\overline{hf})(\kappa) + \eta^{-1}\kappa\bar{f}(\kappa)\tanh\pi\kappa = 0, \tag{2.2.6}$$

and bars denote Fourier transforms. A formal application of the inversion and convolution theorems to equation (2.2.5) yields the result

$$\phi(x,y) = -\pi^{-1}(\sin\tfrac{1}{2}y)\int_{-\infty}^{\infty} f(u)\frac{\cosh\frac{1}{2}(x-u)}{\cosh(x-u)-\cos y}\,\mathrm{d}u, \quad y \in [-\pi, 0), \tag{2.2.7}$$

where

$$\phi(x,0) = f(x), \tag{2.2.8}$$

and their application to equation (2.2.6) yields the expression

$$2\pi^2\eta h(\beta x) f(x) + ⨍_{-\infty}^{\infty} \frac{f'(t)}{\sinh\frac{1}{2}(t-x)}\,\mathrm{d}t = 0. \tag{2.2.9}$$

The symbol $⨍$ means the integral is to be interpreted in its Cauchy principal value sense. In deriving equations (2.2.7) and (2.2.9) the table of Fourier transforms appearing in Lighthill

(1964) and the identities 3.981, 10 and 3.987, 1 of Gradshteyn & Ryzhik (1965) have been used. To find a solution of equation (2.2.9) satisfying the radiation conditions

$$f(x) \to \exp(\mathrm{i}K_{-}x) + \hat{R}\exp(-\mathrm{i}K_{-}x) \quad (x \to -\infty)$$

and
$$f(x) \to \hat{T}\exp(\mathrm{i}K_{+}x) \quad (x \to +\infty) \tag{2.2.10}$$

will be called problem B.

It proves convenient, for both analytical and numerical reasons, to derive an alternative integro-differential equation in which the differentiated term appears outside the integral sign. A simple rearrangement of equation (2.2.6) yields the result

$$\eta^{-1}\kappa\bar{f}(\kappa) - \pi(\overline{hf})(\kappa)(\coth\pi\kappa - (\pi\kappa)^{-1}) - (\pi\eta)^{-1}\bar{f}(\kappa)\tanh\pi\kappa = C_1\delta(\kappa), \tag{2.2.11}$$

where δ is the Dirac δ-function arising as the generalized solution of an equation $\kappa\bar{g}(\kappa) = 0$ and C_1 is an arbitrary constant. A formal application of the inversion and convolution theorems to equation (2.2.11) yields the result

$$f'(x) - \tfrac{1}{2}\eta \int_{-\infty}^{\infty} h(\beta(x+u))\, f(x+u)\,(\coth\tfrac{1}{2}u - \operatorname{sgn} u)\,\mathrm{d}u - \tfrac{1}{2}\pi^{-2}\int_{-\infty}^{\infty} \frac{f(x+u)}{\sinh\tfrac{1}{2}u}\,\mathrm{d}u = C_2, \tag{2.2.12}$$

where the formulae 3.987, 1 and 2 of Gradshteyn & Ryzhik (1965) have been used and C_2 is a constant depending on C_1. The value of C_2 is zero as determined by the behaviour of the function value $f(x)$ for large values of $|x|$ using the identities 3.981, 1 and 3.987, 2 of Gradshteyn & Ryzhik (1965) and the dispersion relations

$$\pi K_{\pm}\tanh\pi K_{\pm} = \pi\eta\begin{Bmatrix}\epsilon\\1\end{Bmatrix}. \tag{2.2.13 a, b}$$

These dispersion relations (2.2.13 *a*, *b*) are consequential to requiring the expressions $\exp(\mathrm{i}K_{\pm}x)$ to be 'local solutions' of the integro-differential equation (2.2.9), that is the results obtained if $h(\beta x)$ assumes the values $h(\pm\infty)$ respectively. To find a solution of equation (2.2.12), satisfying the radiation conditions (2.2.10), will be called problem C.

The above analysis has formally reduced solving problem A for the potential ϕ to the problem of determining the 'wave amplitude' f which solves either problem B or problem C.

THEOREM 1. There exists a one-to-one correspondence between the sets of solutions of problems A, B and C.

Proof. Suppose the function ϕ is defined by equations (2.2.7) and (2.2.8) in terms of a solution f of either problem B or problem C. Then

$$\nabla^2\phi = 0 \quad \text{in} \quad -\infty < x < \infty, \quad -\pi < y < 0$$

and
$$\phi_y = 0 \quad \text{on} \quad y = -\pi.$$

Furthermore, forming the generalized functions corresponding to the ordinary functions ϕ and f, and using the convolution theorem and the identity 3.983, 6 of Gradshteyn & Ryzhik (1965) shows the generalized Fourier transform $\bar{\phi}$ satisfies equation (2.2.5). But as the generalized Fourier transform $\bar{f}$ satisfies either equation (2.2.6) or equation (2.2.11) with the constant $C_1 = 0$, ϕ satisfies the boundary condition (2.2.3). Also, upon applying the radiation conditions (2.2.10) and again using the identity 3.983, 6 of Gradshteyn & Ryzhik (1965), equations (2.2.7) and (2.2.8) require that ϕ satisfies the radiation conditions (2.2.4).

Conversely, suppose the potential ϕ is a solution of problem A. Let the function f be defined by the equation $f(x) = \phi(x, 0)$. The solution ϕ of equations (2.2.1) and (2.2.2) satisfying the radiation conditions (2.2.4) and assuming the value $f(x)$ on the 'free surface' $y = 0$ is given by equation (2.2.7). Thus after forming the generalized functions corresponding to the ordinary functions ϕ and f, using the boundary condition (2.2.3) and following the prior derivation of equation (2.2.11) it follows that f satisfies both the integro-differential equations (2.2.9) and (2.2.12). Clearly the function value $f(x)$ satisfies the radiation conditions (2.2.10).

The correspondence between a solution ϕ of problem A and a solution f of either problem B or problem C is one-to-one since a non-zero f cannot generate a zero contribution to ϕ; for $\phi(x, 0) = f(x)$ and a harmonic function not identically zero on the boundary cannot be the zero function. Also problem B and problem C are equivalent since a solution f of either one of them generates a solution ϕ of problem A via equations (2.2.7) and (2.2.8), but then f is a solution of the remaining problem.

It now follows from equation (2.2.8) that if ϕ is a velocity potential that solves problem A, then the function f is proportional to the wave amplitude ζ. A direct mathematical proof of the existence of a unique solution ψ of the boundary value problem (2.1.1), (2.1.2) and (2.1.3) satisfying the radiation conditions (2.1.5 *a*, *b*) for all bottom profiles b does not seem to exist. However, a proof of the existence of a solution to both problem B and problem C, for general functions h, but restricted to wavelengths either short in comparison with the depth or long in comparison with the transition width, is given in § 3. Thus by theorem 1 this function generates a solution of problem A. Moreover, the respective solutions of problems A, B and C are shown to be unique if the radiation conditions (2.2.4) and (2.2.10) are satisfied with an error of $O(\exp(-\beta|x|))$ as $x \to \pm\infty$, where the rôle of the parameter β has been discussed at the beginning of this subsection.

For convenience in the subsequent analysis, new independent and dependent variables are introduced as follows:

$$X = \beta x$$

and

$$F(X) = f(\beta^{-1}X) = f(x).$$

3. Solution procedures

3.1. *Wavelength small compared with the depth*

The parameter $\sigma = \beta\eta^{-\frac{1}{2}}$ is a measure of the ratio of the wavelength to the transition width. For values of $\sigma \ll 1$, Carrier (1966), using a formal multi-scaling technique, proposed a method of obtaining what is desirably a uniform first approximation to the wave, when the bottom profile b is slowly varying. However, his results are not proven, nor is the reflexion coefficient determined. As his solution is essentially the W.K.B.J. approximation for which Mahony's (1967) results are available, the reflexion coefficient is probably transcendentally small in σ^{-1} so that any numerical process leading to a value of the reflexion coefficient will present formidable difficulties when σ is small.

To discuss the limiting case of values of the parameter $\eta \gg 1$ (and hence $\sigma \ll 1$), the integro-differential equation (2.2.12) is the most useful statement of the problem. The solution is sought in a form motivated by technical considerations

$$F(X) = T(X)\exp\left(\mathrm{i}\sigma^{-2}\int_0^X k(V)\,\mathrm{d}V\right) + R(X)\exp\left(-\mathrm{i}\sigma^{-2}\int_0^X k(V)\,\mathrm{d}V\right), \tag{3.1.1}$$

where the wavenumber $d^{-1}k(X)$ is the real positive root of the dispersion relation

$$\pi\beta\sigma^{-2}k(X)\tanh\pi\beta\sigma^{-2}k(X) = \pi^2\beta^2\sigma^{-2}h(X). \tag{3.1.2}$$

Given the function F, there is no unique decomposition defining the functions T and R. The difficulties associated with a possible lack of uniqueness in problem C are avoided by using this arbitrariness to rearrange the integro-differential equation (2.2.12) into a form that admits of a unique solution. The dispersion relation (3.1.2) is consequential to requiring the expression

$$\exp\left(\mathrm{i}\sigma^{-2}\int_0^X k(V)\,\mathrm{d}V\right)$$

to be a 'local solution' of the integro-differential equation (2.2.12), that is the result obtained if the function value $h(X)$ is assumed constant. For large values of η, the dispersion relation (3.1.2) is conveniently rearranged into the form

$$k(X) = \pi\beta h(X) + 2k(X)\frac{\exp(-2\pi\beta\sigma^{-2}k(X))}{1+\exp(-2\pi\beta\sigma^{-2}k(X))}. \tag{3.1.3}$$

The approximate wavenumber $d^{-1}k(X) \simeq \pi\beta d^{-1}h(X)$ is equivalent to that obtained from Carrier's (1966) dispersion relation involving the bottom profile b.

It is convenient to define the phase function τ by the equation

$$\tau(X) = \int_0^X k(V)\,\mathrm{d}V. \tag{3.1.4}$$

Then substituting equation (3.1.1) into the integro-differential equation (2.2.12) yields the expression

$$\begin{aligned}
&T'(X)\exp(\mathrm{i}\sigma^{-2}\tau(X)) + R'(X)\exp(-\mathrm{i}\sigma^{-2}\tau(X))\\
&\quad = -\mathrm{i}\sigma^{-2}k(X)\,T(X)\exp(\mathrm{i}\sigma^{-2}\tau(X)) + \mathrm{i}\sigma^{-2}k(X)\,R(X)\exp(-\mathrm{i}\sigma^{-2}\tau(X))\\
&\qquad + \tfrac{1}{2}\sigma^{-2}\int_{-\infty}^{\infty} h(X+V)\begin{bmatrix} T(X+V)\exp(\mathrm{i}\sigma^{-2}\tau(X+V))\\ +R(X+V)\exp(-\mathrm{i}\sigma^{-2}\tau(X+V))\end{bmatrix}(\coth\tfrac{1}{2}\beta^{-1}V - \operatorname{sgn} V)\,\mathrm{d}V\\
&\qquad + \tfrac{1}{2}(\pi\beta)^{-2}\int_{-\infty}^{\infty}\begin{bmatrix} T(X+V)\exp(\mathrm{i}\sigma^{-2}\tau(X+V))\\ +R(X+V)\exp(-\mathrm{i}\sigma^{-2}\tau(X+V))\end{bmatrix}\frac{\mathrm{d}V}{\sinh(\frac{1}{2}\beta^{-1}V)}.
\end{aligned} \tag{3.1.5}$$

The problem is now reformulated by separating (3.1.5) into two equations for the functions T and R, which together imply equation (3.1.5), but are not implied by that equation. This separation is achieved by introducing a new function P, related to the functions T and R, by the equation

$$P(X) = T'(X)\exp(\mathrm{i}\sigma^{-2}\tau(X)) = R'(X)\exp(-\mathrm{i}\sigma^{-2}\tau(X)). \tag{3.1.6}$$

To satisfy the radiation conditions (2.2.10), the functions T and R are taken as

$$\left.\begin{aligned}
T(X) &= 1 + \int_{-\infty}^X P(V)\exp(-\mathrm{i}\sigma^{-2}\tau(V))\,\mathrm{d}V,\\
\text{and}\qquad R(X) &= -\int_X^{\infty} P(V)\exp(\mathrm{i}\sigma^{-2}\tau(V))\,\mathrm{d}V,
\end{aligned}\right\} \tag{3.1.7}$$

with the constant factor

$$\sigma^{-2}\int_{-\infty}^{0}(k(-\infty) - k(V))\,\mathrm{d}V$$

absorbed into the parameter Θ. The function P is expressed in terms of integrals of the functions T and R by substituting equation (3.1.6) into equation (3.1.5). The result is given in appendix A.

The motivation behind this choice of the function P is that a solution of the integro-differential equation (2.2.12) is then expressed as the sum of two waves which (to all appearances) travel in opposite directions. Moreover, the functions T and R are the unique solutions of equations (3.1.7) and (A 2). These two equations constitute what is called problem D. For convenience, problem D is expressed in the form of a vector equation

$$\mathscr{F} = \mathscr{A}_0 + \mathscr{M}\mathscr{F}, \tag{3.1.8}$$

where $\mathscr{F}$ is the vector function $(T, R)^{\mathrm{tr}}$, $\mathscr{A}_0$ is the constant vector function $(1, 0)^{\mathrm{tr}}$ and the operator matrix $\mathscr{M}$ is defined in appendix A. The superscript tr stands for transposed.

Let $\mathscr{B}$ be the vector space of C^1 functions $f\colon \mathbb{R}\to\mathbb{C}$ with the property that $f(X)$ and $f'(X)\exp(|X|)$ tend to finite limits as $X\to\pm\infty$. Introduce a norm on the vector space $\mathscr{B}$ as follows

$$\|f\| = \sup_{X\in\mathbb{R}}|f(X)| + \sup_{X\in\mathbb{R}}|f'(X)\exp(|X|)| \quad (\forall f\in\mathscr{B}),$$

and consider the corresponding standard cartesian product norm on the vector space $\mathscr{B}^2 = \mathscr{B}\times\mathscr{B}$. The normed vector space $\mathscr{B}^2$ is then complete.

To show problem D has a unique fixed point in the Banach space $\mathscr{B}^2$, a suitable upper bound for the function P is obtained in the following lemma.

Lemma 1. Suppose the functions $T, R\in\mathscr{B}$. Then there exists a $\sigma_0 > 0$ such that

$$|P(X)| \leqslant K\sigma^2\|(T,R)\|\exp(-|X|) \quad (\forall X\in\mathbb{R}),$$

whenever the parameter $\sigma\in(0,\sigma_0)$, where K is a constant independent of σ.

The proof of lemma 1 is given in appendix B.

Theorem 2. There exists a unique solution of problem D in the vector space $\mathscr{B}^2$ whenever the parameter $\sigma\in(0,\sigma_0^*)$ for some $\sigma_0^* > 0$.

Proof. From lemma 1, a simple calculation yields the inequality

$$\|\mathscr{M}(\mathscr{F}-\mathscr{F}_*)\| \leqslant \hat{K}\sigma^2\|\mathscr{F}-\mathscr{F}_*\|,\ \forall\mathscr{F},\quad \mathscr{F}_*\in\mathscr{B}^2,$$

whenever the parameter $\sigma\in(0,\sigma_0)$. The value of σ_0 is defined in lemma 1 and $\hat{K}$ is a constant independent of σ. Therefore, by the contraction mapping theorem, if

$$\sigma^{-1} > \sigma_0^{*-1} = \max(\hat{K}^{\frac{1}{2}}, \sigma_0^{-1}) > 0$$

then equation (3.1.8) has a unique fixed point $\mathscr{F}_0\in\mathscr{B}^2$.

The matrix product

$$(\exp(\mathrm{i}\sigma^{-2}\tau(X)),\quad \exp(-\mathrm{i}\sigma^{-2}\tau(X)))\,\mathscr{F}_0(X), \tag{3.1.9}$$

defines a solution of problem C for values of the parameter $\sigma\in(0,\sigma_0^*)$. This result is easily seen by retracing the steps in the derivation of equation (3.1.8). It is now shown to be the only solution of the integro-differential equation (2.2.12) satisfying the radiation conditions (2.2.10) with an error of $O(\exp(-|X|))$ as $X\to\pm\infty$.

Lemma 2. If the function $F\in C^1$ satisfies the radiation conditions

$$\left.\begin{aligned} F(X) &= \exp(\mathrm{i}\sigma^{-2}\tau(X)) + r\exp(-\mathrm{i}\sigma^{-2}\tau(X)) + O(\exp(X)) \quad \text{as}\quad X\to-\infty \\ \text{and}\quad F(X) &= t\exp(\mathrm{i}\sigma^{-2}\tau(X)) + O(\exp(-X)) \quad \text{as}\quad X\to+\infty, \end{aligned}\right\} \tag{3.1.10}$$

then there exists unique functions T, $R \in \mathscr{B}$ such that

$$F(X) = T(X)\exp(\mathrm{i}\sigma^{-2}\tau(X)) + R(X)\exp(-\mathrm{i}\sigma^{-2}\tau(X))$$

and
$$T'(X)\exp(\mathrm{i}\sigma^{-2}\tau(X)) = R'(X)\exp(-\mathrm{i}\sigma^{-2}\tau(X)).$$

Proof. The functions T and R are uniquely defined by the equations

$$(T(X)\exp(\mathrm{i}\sigma^{-2}\tau(X)))' = \tfrac{1}{2}(F'(X) + \mathrm{i}\sigma^{-2}k(X)F(X)) \tag{3.1.11}$$

and
$$(R(X)\exp(-\mathrm{i}\sigma^{-2}\tau(X)))' = \tfrac{1}{2}(F'(X) - \mathrm{i}\sigma^{-2}k(X)F(X)), \tag{3.1.12}$$

since the function values $T(X)$ and $R(X)$ have the limiting behaviour $T(X) \to 1$ as $X \to -\infty$ and $R(X) \to 0$ as $X \to +\infty$. Also, as each of the expressions

$$T(X)\exp(\mathrm{i}\sigma^{-2}\tau(X)) \quad \text{and} \quad R(X)\exp(-\mathrm{i}\sigma^{-2}\tau(X))$$

has a continuous derivative at $X = 0$, each of the functions T and R has a continuous derivative there. The formulae for the functions T and R are obtained by integrating equations (3.1.11) and (3.1.12), and using the radiation conditions (3.1.10). The results are

$$(T(X)-1)\exp(\mathrm{i}\sigma^{-2}\tau(X)) = \tfrac{1}{2}F_-(X) + \tfrac{1}{2}\mathrm{i}\sigma^{-2}\int_{-\infty}^{X} k(V)F_-(V)\,\mathrm{d}V, \quad (\forall X \leqslant 0),$$

$$(T(X)-t)\exp(\mathrm{i}\sigma^{-2}\tau(X)) = \tfrac{1}{2}F_+(X) - \tfrac{1}{2}\mathrm{i}\sigma^{-2}\int_{X}^{\infty} k(V)F_+(V)\,\mathrm{d}V, \quad (\forall X \geqslant 0),$$

$$(R(X)-r)\exp(-\mathrm{i}\sigma^{-2}\tau(X)) = \tfrac{1}{2}F_-(X) - \tfrac{1}{2}\mathrm{i}\sigma^{-2}\int_{-\infty}^{X} k(V)F_-(V)\,\mathrm{d}V \quad (\forall X \leqslant 0),$$

and
$$R(X)\exp(-\mathrm{i}\sigma^{-2}\tau(X)) = \tfrac{1}{2}F_+(X) + \tfrac{1}{2}\mathrm{i}\sigma^{-2}\int_{X}^{\infty} k(V)F_+(V)\,\mathrm{d}V \quad (\forall X \geqslant 0),$$

where the functions $F_\pm$ are defined by the equations

$$F_+(X) = F(X) - t\exp(\mathrm{i}\sigma^{-2}\tau(X))$$

and
$$F_-(X) = F(X) - \exp(\mathrm{i}\sigma^{-2}\tau(X)) - r\exp(-\mathrm{i}\sigma^{-2}\tau(X)).$$

Corollary. The constants r, $t \in \mathbb{C}$ satisfy the identity

$$1 - r - t + \mathrm{i}\sigma^{-2}\int_{0}^{\infty} k(V)F_+(V)\,\mathrm{d}V + \mathrm{i}\sigma^{-2}\int_{-\infty}^{0} k(V)F_-(V)\,\mathrm{d}V = 0.$$

Proof. This result follows immediately from the continuity of the functions T and R at the origin $X = 0$.

THEOREM 3. There exists a unique C^1 solution of the integro-differential equation (2.2.12) that satisfies the radiation conditions (3.1.10) whenever the parameter $\sigma \in (0, \sigma_0^*)$.

Proof. The matrix product (3.1.9) defines one such solution. Suppose the functions F_1 and F_2 are two solutions. Then by lemma 2 there exists unique vector functions $\mathscr{F}_1$, $\mathscr{F}_2 \in \mathscr{B}^2$ in terms of which the functions F_1 and F_2 are defined by the matrix products

$$F_j(X) = (\exp(\mathrm{i}\sigma^{-2}\tau(X)),\ \exp(-\mathrm{i}\sigma^{-2}\tau(X)))\,\mathscr{F}_j(X) \quad (j = 1, 2)$$

and which satisfy the matrix product identities

$$(\exp(\mathrm{i}\sigma^{-2}\tau(X)),\ -\exp(-\mathrm{i}\sigma^{-2}\tau(X)))\,\mathscr{F}_j'(X) = 0 \quad (j = 1, 2).$$

But then by theorem 2, $\mathscr{F}_1 = \mathscr{F}_2$ whenever $\sigma \in (0, \sigma_0^*)$ since both are solutions of problem D. Hence $F_1 = F_2$ and the integro-differential equation (2.2.12) has a unique solution.

Asymptotic nature of the solution

Consider the Picard iteration scheme defining the nth iterate $\mathscr{F}^{(n)}$ by the recursion formula

$$\mathscr{F}^{(n)} = \mathscr{A}_0 + \mathscr{M}\mathscr{F}^{(n-1)} \quad (n \geqslant 1), \tag{3.1.13}$$

where the initial iterate $\mathscr{F}^{(0)}$ takes the value $\mathscr{A}_0$. The Picard iteration scheme (3.1.13) converges to the unique fixed point $\mathscr{F}_0 \in \mathscr{B}^2$ and it can be shown in the standard way that the matrix product

$$(\exp(\mathrm{i}\sigma^{-2}\tau(X)),\ \exp(-\mathrm{i}\sigma^{-2}\tau(X)))\ (\mathscr{F}_0(X) - \mathscr{F}^{(N)}(X)),$$

is $O(\sigma^{N+1})$ as the parameter $\sigma \to 0$. Thus an asymptotic expansion $\hat{\mathscr{F}}^{(N)}$ of the Nth iterate $\mathscr{F}^{(N)}$ (or equivalently the component functions $T^{(N)}$ and $R^{(N)}$) will yield an asymptotic expansion of $\mathscr{F}_0$ to N terms. The matrix product

$$(\exp(\mathrm{i}\sigma^{-2}\tau(X)),\ \exp(-\mathrm{i}\sigma^{-2}\tau(X)))\ \hat{\mathscr{F}}^{(N)}(X),$$

is then an asymptotic estimate of the solution (3.1.9) of problem C in the limiting case as $\sigma \to 0$.

A first approximation to the 'reflective' part of the solution (3.1.9) is given by the formula

$$R^{(1)}(X) = -\sigma \int_X^\infty P_0(V) \exp(2\mathrm{i}\sigma^{-2}\tau(V))\,\mathrm{d}V. \tag{3.1.14}$$

The function $X \mapsto P_0(X)\exp(\mathrm{i}\sigma^{-2}\tau(X))$ is defined by equation (A 2) in the particular case when the functions T and R are the constant functions 1 and 0 respectively, that is

$$P_0(X)\exp(\mathrm{i}\sigma^{-2}\tau(X)) = \sigma^{-1}P(X)|_{T=1,\,R=0}.$$

Applying Taylor's theorem and then replacing the function value $\hat{\theta}(X, w)$ by the approximation $w(k(X))^{-1}$ yields the formal estimate

$$P_0(X) \simeq 2(\pi\beta)^3 (hk')(X)\,\sigma^{-5}\exp(-2\pi\beta k(X)\,\sigma^{-2}).$$

Thus, after one integration by parts, a formal approximation to the first iterate (3.1.14) is given by the expression

$$R^{(1)}(X) \simeq -(\pi\beta)^3\sigma^{-2}(hk')(X)\frac{\pi\beta k'(X) + \mathrm{i}k(X)}{\pi^2\beta^2(k'(X))^2 + k^2(X)}\exp(2(-\pi\beta k(X) + \mathrm{i}\tau(X))\,\sigma^{-2})$$
$$+ (\pi\beta)^3\sigma^{-2}\int_X^\infty \left(\frac{(hk')(V)}{-\pi\beta k'(V) + \mathrm{i}k(V)}\right)' \exp(2(-\pi\beta k(V) + \mathrm{i}\tau(V))\,\sigma^{-2})\,\mathrm{d}V.$$

Discussion

The approximation developed by Carrier (1966) agrees with the solution (3.1.9) only in the limiting case as $\sigma \to 0$, for each fixed value of β. The wavelengths determined by the two methods are not equivalent for non-zero values of σ. In Carrier's (1966) method the local value of the wavelength is determined from the local value of the depth. However, the local value of the wavelength is determined by the global features of the bottom profile b; this dependence being achieved with the function h.

The solution (3.1.9) of problem C is expressible in the form of the sum of a 'transmissive' and a 'reflective' term. The reflexion coefficient $|r|^2$ is defined by the equation

$$|r|^2 = |R(-\infty)|^2 = \left|\int_{-\infty}^\infty P(V)\exp(\mathrm{i}\sigma^{-2}\tau(V))\,\mathrm{d}V\right|^2,$$

where an approximation to the value $R(-\infty)$ is given by equation (3.1.14) as

$$R(-\infty) \simeq -\sigma \int_{-\infty}^{\infty} P_0(V) \exp\left(2\mathrm{i}\sigma^{-2}\tau(V)\right) \mathrm{d}V + o(\sigma). \tag{3.1.15}$$

To accurately estimate the integral $\int_{-\infty}^{\infty} \exp\left(\ln|P_0(V)| + 2\mathrm{i}\sigma^{-2}\tau(V)\right) \mathrm{d}V$, the cols of the expression $\ln|P_0(V)| + 2\mathrm{i}\sigma^{-2}\tau(V)$ should be located and the method of steepest descents applied.

3.2. *Wavelength large compared with the transition width*

In the limiting case when the wavelength is much greater than the depth, Lamb estimated the reflexion coefficient, for a bottom profile in which there is an abrupt change of depth, by matching the limiting wave forms across the discontinuity. A rigorous justification of this result is given by Bartholomeusz (1958) in the special case of long waves incident on a step. In Kajiura (1963), Lamb's approximation is applied to bottom profiles that do not possess abrupt changes in depth and tentative bounds are given for the applicability of the results. Here this approximation is shown to be a valid one for general bottom profiles b for the ratio of wavelength to transition width $\gg 1$.

To discuss the limiting case $\sigma \gg 1$, the linear combination of the integro-differential equations (2.2.9) and (2.2.12)

$$F''(X) = \tfrac{1}{2}\sigma^{-2} \fint_{-\infty}^{\infty} \left(h(X+V)\, F(X+V)\right)' \left(\coth \tfrac{1}{2}\beta^{-1}V - \operatorname{sgn} V\right) \mathrm{d}V - \sigma^{-2} h(X)\, F(X), \tag{3.2.1}$$

is the most useful statement of the problem. As in § 3.1, the solution is sought in a form motivated by technical considerations

$$F(X) = \begin{cases} T_+(X) \exp\left(\mathrm{i}\sigma^{-1}k_+ X\right) + R_+(X) \exp\left(-\mathrm{i}\sigma^{-1}k_+ X\right) & (\forall X \geqslant 0), \\ T_-(X) \exp\left(\mathrm{i}\sigma^{-1}k_- X\right) + R_-(X) \exp\left(-\mathrm{i}\sigma^{-1}k_- X\right) & (\forall X \leqslant 0), \end{cases} \tag{3.2.2}$$

where the wavenumbers $d^{-1}k_\pm$ are the real positive roots of the dispersion relations

$$\pi\beta\sigma^{-1}k_\pm \tanh \pi\beta\sigma^{-1}k_\pm = \pi^2\beta^2\sigma^{-2}h(\pm\infty). \tag{3.2.3 a, b}$$

Here the difficulties associated with a possible lack of uniqueness in problems B and C are avoided by rearranging the differential-integral equation (3.2.1) into a form that admits of a unique solution. The dispersion relations (3.2.3 a, b) are consequential to requiring the expressions $\exp(\mathrm{i}\sigma^{-1}k_\pm X)$ to be 'local solutions' of either integro-differential equation (2.2.9) or (2.2.12), that is the results obtained if $h(X)$ assumes the values $h(\pm\infty)$ respectively. For large values of σ the dispersion relations (3.2.3 a, b) are conveniently rearranged into the forms

$$k_\pm^2 = h(\pm\infty) + (\pi\beta)^{-1}\sigma k_\pm (\pi\beta\sigma^{-1}k_\pm - \tanh \pi\beta\sigma^{-1}k_\pm). \tag{3.2.4 a, b}$$

Upon substituting equation (3.2.2) into the differential-integral equation (3.2.1) yields the expressions

$$\begin{aligned} &\left(T_\pm''(X) + 2\mathrm{i}\sigma^{-1}k_\pm T_\pm'(X)\right) \exp\left(\mathrm{i}\sigma^{-1}k_\pm X\right) + \left(R_\pm''(X) - 2\mathrm{i}\sigma^{-1}k_\pm R_\pm'(X)\right) \exp\left(-\mathrm{i}\sigma^{-1}k_\pm X\right) \\ &\quad = \tfrac{1}{2}\sigma^{-2} \fint_{-\infty}^{\infty} \left(h(X+V)\, F(X+V)\right)' \left(\coth \tfrac{1}{2}\beta^{-1}V - \operatorname{sgn} V\right) \mathrm{d}V \\ &\qquad - \sigma^{-2}\left(h(X) - k_\pm^2\right) \left(T_\pm(X) \exp\left(\mathrm{i}\sigma^{-1}k_\pm X\right) + R_\pm(X) \exp\left(-\mathrm{i}\sigma^{-1}k_\pm X\right)\right). \end{aligned} \tag{3.2.5 a, b}$$

In this limiting case the problem is reformulated by separating equations (3.2.5 a, b) into four equations for the functions $T_\pm$ and $R_\pm$, which together imply equations (3.2.5 a, b), but are not implied by that equation. The separation is achieved by introducing two new functions $G_\pm$, related to the functions $T_\pm$ and $R_\pm$, by the equations

$$G_\pm(X) = 2\mathrm{i}\sigma k_\pm T'_\pm(X)\exp(\mathrm{i}\sigma^{-1}k_\pm X) = -2\mathrm{i}\sigma k_\pm R'_\pm(X)\exp(-\mathrm{i}\sigma^{-1}k_\pm X). \qquad (3.2.6\,a, b)$$

To satisfy the radiation conditions (2.2.10) and the requirements that the function F has a continuous derivative at the origin $X = 0$, the functions $T_\pm$ and $R_\pm$ are taken as

$$T_+(X) = \frac{2k_-}{k_+ + k_-} - \tfrac{1}{2}\mathrm{i}\sigma^{-1}k_+^{-1}\int_0^X G_+(V)\exp(-\mathrm{i}\sigma^{-1}k_+V)\,\mathrm{d}V - \tfrac{1}{2}\mathrm{i}\frac{k_+ - k_-}{k_+ + k_-}\sigma^{-1}k_+^{-1}\int_0^\infty G_+(V)\exp(\mathrm{i}\sigma^{-1}k_+V)\,\mathrm{d}V - \frac{\mathrm{i}\sigma^{-1}}{k_+ + k_-}\int_{-\infty}^0 G_-(V)\exp(-\mathrm{i}\sigma^{-1}k_-V)\,\mathrm{d}V, \qquad (3.2.7)$$

$$R_+(X) = -\tfrac{1}{2}\mathrm{i}\sigma^{-1}k_+^{-1}\int_X^\infty G_+(V)\exp(\mathrm{i}\sigma^{-1}k_+V)\,\mathrm{d}V, \qquad (3.2.8)$$

$$T_-(X) = 1 - \tfrac{1}{2}\mathrm{i}\sigma^{-1}k_-^{-1}\int_{-\infty}^X G_-(V)\exp(-\mathrm{i}\sigma^{-1}k_-V)\,\mathrm{d}V \qquad (3.2.9)$$

and

$$R_-(X) = \frac{k_- - k_+}{k_- + k_+} - \tfrac{1}{2}\mathrm{i}\sigma^{-1}k_-^{-1}\int_X^0 G_-(V)\exp(\mathrm{i}\sigma^{-1}k_-V)\,\mathrm{d}V - \tfrac{1}{2}\mathrm{i}\frac{k_- - k_+}{k_- + k_+}\sigma^{-1}k_-^{-1}\int_{-\infty}^0 G_-(V)\exp(-\mathrm{i}\sigma^{-1}k_-V)\,\mathrm{d}V - \frac{\mathrm{i}\sigma^{-1}}{k_+ + k_-}\int_0^\infty G_+(V)\exp(\mathrm{i}\sigma^{-1}k_+V)\,\mathrm{d}V. \qquad (3.2.10)$$

The functions $G_\pm$ are expressed in terms of the underived functions F, $T_\pm$ and $R_\pm$ by substituting equations (3.2.6 a, b) into equations (3.2.5 a, b). The resulting expressions for the functions $G_\pm$ are given in appendix C.

To find a solution of equations (3.2.7)–(3.2.10), C1 and C2 is called problem E. Again for convenience, problem E is expressed in the form of a vector equation

$$\mathscr{F} = \mathscr{A}_0 + \sigma^{-1}\mathscr{M}\mathscr{F}, \qquad (3.2.11)$$

where $\mathscr{F}$ is the vector function $(T_+, T_-, R_+, R_-)^{\mathrm{tr}}$, $\mathscr{A}_0$ is the constant vector function $(2k_-(k_+ + k_-)^{-1}, 1, 0, (k_- - k_+)(k_+ + k_-)^{-1})^{\mathrm{tr}}$ and the operator matrix $\mathscr{M}$ is defined in appendix C. The superscript tr again stands for transposed.

Let $\mathscr{B}^\pm$ be the vector spaces of C^1 functions f^+: $[0, +\infty) \to \mathbb{C}$ and f^-: $(-\infty, 0] \to \mathbb{C}$ respectively, with the property that $f^\pm(X)$ and $(f^\pm)'(X)\exp(\tfrac{1}{2}|X|)$ tend to finite limits as $X \to \pm\infty$ respectively. Introduce norms on the vector spaces $\mathscr{B}^\pm$ as follows

$$\|f^\pm\| = \sup_{X\in\operatorname{dom}f^\pm}|f^\pm(X)| + \sup_{X\in\operatorname{dom}f^\pm}|(f^\pm)'(X)\exp(\tfrac{1}{2}|X|)| \quad (\forall f^\pm \in \mathscr{B}^\pm),$$

and consider the corresponding standard cartesian product norm on the vector space $(\mathscr{B}^+\times\mathscr{B}^-)^2 = \mathscr{B}^+\times\mathscr{B}^-\times\mathscr{B}^+\times\mathscr{B}^-$. The normed vector space $(\mathscr{B}^+\times\mathscr{B}^-)^2$ is then complete.

If we proceed as in § 3.1, to show that problem E has a unique fixed point in the Banach space $(\mathscr{B}^+\times\mathscr{B}^-)^2$, suitable upper bounds for the functions $G_\pm$ are obtained in the following lemma.

Lemma 3. Suppose the functions $T_\pm$, $R_\pm \in \mathscr{B}^\pm$. Then there exists a $\boldsymbol{\sigma}_0 > 0$ such that

$$|G_\pm(X)| \leqslant \boldsymbol{K}\|(T_+, T_-, R_+, R_-)\| \exp(-\tfrac{1}{2}|X|) \quad (\forall X \gtrless 0),$$

whenever the parameter $\sigma > \boldsymbol{\sigma}_0$, where $\boldsymbol{K}$ is a constant independent of σ.

The proof of lemma 3 is given in appendix D.

Theorem 4. There exists a unique solution of problem E in the vector space $(\mathscr{B}^+ \times \mathscr{B}^-)^2$ whenever the parameter $\sigma > \boldsymbol{\sigma}_0^*$ for some $\boldsymbol{\sigma}_0^* > 0$.

Proof. From lemma 3, a simple calculation yields the inequality

$$\|\sigma^{-1}\mathscr{M}(\mathscr{F} - \mathscr{F}_*)\| \leqslant \hat{\boldsymbol{K}}\sigma^{-1}\|\mathscr{F} - \mathscr{F}_*\|, \quad \forall \mathscr{F}, \mathscr{F}_* \in (\mathscr{B}^+ \times \mathscr{B}^-)^2,$$

whenever $\sigma > \boldsymbol{\sigma}_0$. The value of $\boldsymbol{\sigma}_0$ is defined by lemma 3 and $\hat{\boldsymbol{K}}$ is a constant independent of σ. Therefore, by the contraction mapping theorem, if $\sigma > \boldsymbol{\sigma}_0^* = \max(\hat{\boldsymbol{K}}, \boldsymbol{\sigma}_0) > 0$ then equation (3.2.11) has a unique fixed point $\mathscr{F}_0 \in (\mathscr{B}^+ \times \mathscr{B}^-)^2$.

The function defined by the matrix products

$$X \mapsto \begin{cases} (\exp(\mathrm{i}\sigma^{-1}k_+X),\ 0,\ \exp(-\mathrm{i}\sigma^{-1}k_+X),\ 0)\,\mathscr{F}_0(X) & (\forall X \geqslant 0) \\ (0,\ \exp(\mathrm{i}\sigma^{-1}k_-X),\ 0,\ \exp(-\mathrm{i}\sigma^{-1}k_-X))\,\mathscr{F}_0(X) & (\forall X \leqslant 0) \end{cases} \tag{3.2.12}$$

is a solution of the differential-integral equation (3.2.1) for values of the parameter $\sigma > \boldsymbol{\sigma}_0^*$ that satisfies the radiation conditions (2.2.10). This result is easily seen by retracing the steps in the derivation of equation (3.2.11). It is now shown to be the only solution of the differential-integral equation (3.2.1) that satisfies the radiation conditions (2.2.10) with an error of $O(\exp(-\tfrac{1}{2}|X|))$ as $X \to \pm\infty$.

Lemma 4. If the function $F \in C^2$ satisfies the radiation conditions

$$\left.\begin{aligned} F(X) &= \exp(\mathrm{i}\sigma^{-1}k_-X) + \boldsymbol{r}\exp(-\mathrm{i}\sigma^{-1}k_-X) + O(\exp(\tfrac{1}{2}X)) \quad \text{as} \quad X \to -\infty \\ \text{and} \quad F(X) &= \boldsymbol{t}\exp(\mathrm{i}\sigma^{-1}k_+X) + O(\exp(-\tfrac{1}{2}X)) \quad \text{as} \quad X \to +\infty, \end{aligned}\right\} \tag{3.2.13}$$

then there exists unique functions $T_\pm$, $R_\pm \in \mathscr{B}^\pm$ such that

$$F(X) = \begin{cases} T_+(X)\exp(\mathrm{i}\sigma^{-1}k_+X) + R_+(X)\exp(-\mathrm{i}\sigma^{-1}k_+X) & (\forall X \geqslant 0) \\ T_-(X)\exp(\mathrm{i}\sigma^{-1}k_-X) + R_-(X)\exp(-\mathrm{i}\sigma^{-1}k_-X) & (\forall X \leqslant 0) \end{cases}$$

and

$$T'_\pm(X)\exp(\mathrm{i}\sigma^{-1}k_\pm X) + R'_\pm(X)\exp(-\mathrm{i}\sigma^{-1}k_\pm X) = 0.$$

Proof. The functions $T_\pm$ and $R_\pm$ are defined by the equations

$$T_\pm(X) = -\tfrac{1}{2}\mathrm{i}\sigma k_\pm^{-1}(F'(X) + \mathrm{i}\sigma^{-1}k_\pm F(X))\exp(-\mathrm{i}\sigma^{-1}k_\pm X)$$

and

$$R_\pm(X) = \tfrac{1}{2}\mathrm{i}\sigma k_\pm^{-1}(F'(X) - \mathrm{i}\sigma^{-1}k_\pm F(X))\exp(\mathrm{i}\sigma^{-1}k_\pm X).$$

Theorem 5. There exists a unique C^2 solution of the differential-integral equation (3.2.1) that satisfies the radiation conditions (3.2.13) whenever the parameter $\sigma > \boldsymbol{\sigma}_0^*$.

Proof. The function defined by the matrix products (3.2.12) is one such solution. Suppose the functions $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are two solutions. Then by lemma 4 there exists unique vector functions $\mathscr{F}_1, \mathscr{F}_2 \in (\mathscr{B}^+ \times \mathscr{B}^-)^2$ in terms of which the functions $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are defined by the matrix products

$$\boldsymbol{F}_j(X) = \begin{cases} (\exp(\mathrm{i}\sigma^{-1}k_+X),\ 0,\ \exp(-\mathrm{i}\sigma^{-1}k_+X),\ 0)\,\mathscr{F}_j(X) & (\forall X \geqslant 0) \\ (0,\ \exp(\mathrm{i}\sigma^{-1}k_-X),\ 0,\ \exp(-\mathrm{i}\sigma^{-1}k_-X))\,\mathscr{F}_j(X) & (\forall X \leqslant 0) \end{cases} \quad (j = 1, 2)$$

and which satisfy the matrix product identities

$$\left.\begin{aligned} &(\exp(\mathrm{i}\sigma^{-1}k_+X),\ 0,\ \exp(-\mathrm{i}\sigma^{-1}k_+X),\ 0)\,\mathscr{F}_j'(X) = 0\\ \text{and}\qquad &(0,\exp(\mathrm{i}\sigma^{-1}k_-X),\ 0,\ \exp(-\mathrm{i}\sigma^{-1}k_-X))\,\mathscr{F}_j'(X) = 0\end{aligned}\right\}\quad (j=1,2).$$

But then by theorem 4 $\mathscr{F}_1 = \mathscr{F}_2$ whenever $\sigma > \sigma_0^*$, since both are solutions of problem E. Hence $\boldsymbol{F}_1 = \boldsymbol{F}_2$ and the differential-integral equation (3.2.1) has a unique solution.

Asymptotic nature of the solution

Consider the Picard iteration scheme defining the nth iterate $\mathscr{F}^{(n)}$ by the recursion formula

$$\mathscr{F}^{(n)} = \mathscr{A}_0 + \sigma^{-1}\mathscr{M}\mathscr{F}^{(n-1)} \quad (n \geqslant 1), \tag{3.2.14}$$

where the initial iterate $\mathscr{F}^{(0)}$ takes the value $\mathscr{A}_0$. The Picard iteration scheme (3.2.14) converges to the unique fixed point $\mathscr{F}_0 \in (\mathscr{B}^+ \times \mathscr{B}^-)^2$ and it can be shown in the standard way that the function defined by the matrix products

$$X \mapsto \begin{cases}(\exp(\mathrm{i}\sigma^{-1}k_+X),\ 0,\ \exp(-\mathrm{i}\sigma^{-1}k_+X),\ 0)\,(\mathscr{F}_0(X) - \mathscr{F}^{(N)}(X)), & \forall X \geqslant 0,\\ (0,\ \exp(\mathrm{i}\sigma^{-1}k_-X),\ 0,\ \exp(-\mathrm{i}\sigma^{-1}k_-X))\,(\mathscr{F}_0(X) - \mathscr{F}^{(N)}(X)), & \forall X \leqslant 0,\end{cases}$$

is $O(\sigma^{-N-1})$ as the parameter $\sigma \to +\infty$. Thus, here also, an asymptotic expansion $\hat{\mathscr{F}}^{(N)}$ of the Nth iterate $\mathscr{F}^{(N)}$ (or equivalently the component functions $T_\pm^{(N)}$ and $R_\pm^{(N)}$) will yield an asymptotic expansion of $\mathscr{F}_0$ to N terms. The function defined by the matrix products

$$X \mapsto \begin{cases}(\exp(\mathrm{i}\sigma^{-1}k_+X),\ 0,\ \exp(-\mathrm{i}\sigma^{-1}k_+X),\ 0)\,\hat{\mathscr{F}}^{(N)}(X), & \forall X \geqslant 0,\\ (0,\ \exp(\mathrm{i}\sigma^{-1}k_-X),\ 0,\ \exp(-\mathrm{i}\sigma^{-1}k_-X))\,\hat{\mathscr{F}}^{(N)}(X), & \forall X \leqslant 0\end{cases}$$

is then an asymptotic estimate of the solution (3.2.12) of the differential-integral equation (3.2.1) in the limiting case as $\sigma \to +\infty$.

A first approximation to the 'reflective' part of the solution (3.2.12) is obtained from equation (3.2.10). The result is Lamb's approximation

$$R_-^{(0)}(X) = \frac{k_- - k_+}{k_- + k_+},$$

valid for the range of the parameter $\sigma \gg 1$. In the particular situation of the depths at $X = +\infty$ and $X = -\infty$ being the same, the first non-zero contribution to the 'reflective' part of the solution (3.2.12) is given by the formula

$$R^{(1)}(X) \simeq -\tfrac{1}{2}\mathrm{i}\sigma^{-1}(h(\infty))^{-\frac{1}{2}}\int_X^\infty G^{(1)}(V)\,\mathrm{d}V,$$

where the function $G^{(1)}$ is defined by the equation

$$G^{(1)}(X) = \frac{1}{2}\int_{-\infty}^{\infty}(h'(V) - h'(X))\,(\coth\tfrac{1}{2}\beta^{-1}(V-X) - \operatorname{sgn}(V-X))\,\mathrm{d}V - (h(X) - h(\infty)).$$

Discussion

The approximation suggested by Lamb agrees with the solution (3.2.12) in the limiting case as $\sigma \to +\infty$. For values of σ not large the reflexion coefficient $|\boldsymbol{r}|^2$ is defined by the equation

$$\begin{aligned}|\boldsymbol{r}|^2 &= |R_-(-\infty)|^2\\ &= \left|\frac{k_- - k_+}{k_- + k_+} - \tfrac{1}{2}\mathrm{i}\sigma^{-1}k_-^{-1}\int_{-\infty}^0 G_-(V)\exp(\mathrm{i}\sigma^{-1}k_-V)\,\mathrm{d}V\right.\\ &\quad\left. - \tfrac{1}{2}\mathrm{i}\frac{k_- - k_+}{k_- + k_+}\sigma^{-1}k_-^{-1}\int_{-\infty}^0 G_-(V)\exp(-\mathrm{i}\sigma^{-1}k_-V)\,\mathrm{d}V - \frac{\mathrm{i}\sigma^{-1}}{k_+ + k_-}\int_0^\infty G_+(V)\exp(\mathrm{i}\sigma^{-1}k_+V)\,\mathrm{d}V\right|^2.\end{aligned}$$

The Picard iteration scheme (3.2.14) provides an explicit technique for obtaining accurate estimates of the reflexion coefficient for values of the parameter σ not all that large.

3.3. *Numerical iteration*

In this section a numerical method of determining the reflexion coefficient is developed for use when the parameter η is of unit order.

Difficulties associated with the oscillatory behaviour of the solution as $X \to \pm\infty$ are overcome by partitioning the domain of the independent variable into the three intervals $(-\infty, -R_0]$, $(-R_0, R_1)$ and $[R_1, +\infty)$. The constants R_0 and R_1 are chosen sufficiently large to ensure that, in the intervals $(-\infty, -R_0]$ and $[R_1, +\infty)$, h is a slowly varying function and the solution is approximated by the leading order terms of the iteration scheme developed in § 3.1. The optimum values of R_0 and R_1 are those for which a small number of iterates accurately determines the solution in the intervals $(-\infty, -R_0]$ and $[R_1, +\infty)$, and a suitable continuous linear operator, in terms of which the wave is defined in the interval $(-R_0, R_1)$, is approximated by not too large a number of well-conditioned algebraic equations.

Asymptotic solution in $(-\infty, -R_0]$ *and* $[R_1, +\infty)$

In the intervals $(-\infty, -R_0]$ and $[R_1, +\infty)$ the solution is determined iteratively; the number of iterates chosen to ensure the approximation is uniformly asymptotic to a specified order. Let $\hat{F}_\pm$ be the restriction of the function F to the intervals $[R_1, +\infty)$ and $(-\infty, -R_0]$ respectively and define the phase functions $\tau_\pm$ by the equations

$$\tau_+(X) = \int_{R_1}^{X} k(V)\,\mathrm{d}V \quad \text{for} \quad X \geqslant R_1, \tag{3.3.1}$$

and

$$\tau_-(X) = \int_{-R_0}^{X} k(V)\,\mathrm{d}V \quad \text{for} \quad X \leqslant -R_0. \tag{3.3.2}$$

Then a solution in the intervals $(-\infty, -R_0]$ and $[R_1, +\infty)$ having a wave of unit amplitude travelling to the right at $X = -\infty$ and no wave travelling to the left at $X = +\infty$ is given by the formulae

$$\hat{F}_\pm(X) = \hat{T}_\pm(X) \exp(\mathrm{i}\sigma^{-2}\tau_\pm(X)) + \hat{R}_\pm(X) \exp(-\mathrm{i}\sigma^{-2}\tau_\pm(X)), \tag{3.3.3 a, b}$$

where the functions $\hat{T}_\pm$ and $\hat{R}_\pm$ are defined by the equations

$$\hat{T}_+(X) = \hat{A}_+ + \tfrac{1}{2}\sigma^{-2}\int_{R_1}^{X} k(V)\,g_+(V)\exp(-\mathrm{i}\sigma^{-2}\tau_+(V))\,\mathrm{d}V, \tag{3.3.4 a}$$

$$\hat{R}_+(X) = -\tfrac{1}{2}\sigma^{-2}\int_{X}^{\infty} k(V)\,l_+(V)\exp(\mathrm{i}\sigma^{-2}\tau_+(V))\,\mathrm{d}V, \tag{3.3.4 b}$$

$$\hat{T}_-(X) = \exp\left(-\mathrm{i}\sigma^{-2}\int_{-\infty}^{-R_0}(k(-\infty)-k(V))\,\mathrm{d}V\right) + \tfrac{1}{2}\sigma^{-2}\int_{-\infty}^{X} k(V)\,g_-(V)\exp(-\mathrm{i}\sigma^{-2}\tau_-(V))\,\mathrm{d}V \tag{3.3.4 c}$$

and

$$\hat{R}_-(X) = \hat{B}_- - \tfrac{1}{2}\sigma^{-2}\int_{X}^{-R_0} k(V)\,l_-(V)\exp(\mathrm{i}\sigma^{-2}\tau_-(V))\,\mathrm{d}V. \tag{3.3.4 d}$$

The functions $g_\pm$ and $l_\pm$ arise in a similar way to the function P of § 3.1 and the constants $\hat{A}_+$ and $\hat{B}_-$ are specified by the requirement that the function F has a continuous derivative at $X = -R_0$ and $X = R_1$; the matching of the functions F and $\hat{F}_-$ at $X = -R_0$ and of the functions F and $\hat{F}_+$ at $X = R_1$ is performed analytically.

A functional form for F in $(-R_0, R_1)$

To permit the matching at $X = -R_0$ and $X = R_1$, the integro-differential equation (2.2.12) is written as

$$F'(X) - \tfrac{1}{2}\sigma^{-2}h(X) \fint_{-\infty}^{\infty} \frac{F(X+V)}{\sinh \frac{1}{2}\beta^{-1}V} \mathrm{d}V - \tfrac{1}{2}(\pi\beta)^{-2} \fint_{-\infty}^{\infty} \frac{F(X+V)}{\sinh \frac{1}{2}\beta^{-1}V} \mathrm{d}V$$

$$-\tfrac{1}{2}\sigma^{-2} \int_{-\infty}^{\infty} F(X+V) \left(h(X+V) \coth \tfrac{1}{2}\beta^{-1}V - \frac{h(X)}{\sinh \frac{1}{2}\beta^{-1}V} - h(X+V) \operatorname{sgn} V \right) \mathrm{d}V = 0. \tag{3.3.5}$$

Differentiating this equation with respect to X, inverting an integration and differentiation and using the integro-differential equation (2.2.9) leads to the result

$$(h(X))^{-1}((h(X))^{-1}F'(X))' + \pi^2\eta^2\beta^{-2}F(X)$$
$$= h(X) \left(\tfrac{1}{2}\sigma^{-2}(h(X))^{-1} \fint_{-\infty}^{\infty} F(X+V)\, h(X+V)\, (\coth \tfrac{1}{2}\beta^{-1}V - \operatorname{sgn} V)\, \mathrm{d}V \right.$$
$$\left. + \tfrac{1}{2}((\pi\beta)^{-2}(h(X))^{-1} - \sigma^{-2}) \fint_{-\infty}^{\infty} \frac{F(X+V)}{\sinh \frac{1}{2}\beta^{-1}V} \mathrm{d}V \right)'. \tag{3.3.6}$$

The formal solution of this equation is

$$F(X) = \bar{C} \exp(\mathrm{i}\tau_0(X)) + \bar{D} \exp(-\mathrm{i}\tau_0(X)) + \int_0^X \varphi(V) \cos(\tau_0(X) - \tau_0(V))\, \mathrm{d}V, \tag{3.3.7}$$

where the functions τ_0 and φ are defined by the equations

$$\tau_0(X) = \varpi \int_0^X h(V)\, \mathrm{d}V, \tag{3.3.8}$$

$$\varphi(X) = \tfrac{1}{2}\sigma^{-2} \fint_{-\infty}^{\infty} F(X+V)\, h(X+V)\, (\coth \tfrac{1}{2}\beta^{-1}V - \operatorname{sgn} V)\, \mathrm{d}V$$
$$+ \tfrac{1}{2}((\pi\beta)^{-2} - \sigma^{-2}h(X)) \fint_{-\infty}^{\infty} \frac{F(X+V)}{\sinh \frac{1}{2}\beta^{-1}V} \mathrm{d}V, \tag{3.3.9}$$

the constant $\varpi = \pi\eta\beta^{-1}$ and $\bar{C}$, $\bar{D}$ are arbitrary constants. This formal solution (3.3.7) enables the desired matching to be performed.

Determination of the constants

The constants R_0 and R_1 are now chosen sufficiently large for the zero approximations to the functions $\hat{F}_{\pm}$ to be suitable estimates of the solution in the intervals $[R_1, +\infty)$ and $(-\infty, -R_0]$, and for the function value $h(X)$ to be well approximated by the constants $h_{\pm} = h(\pm\infty)$. The dispersion relation (3.1.2) then implies that the function value

$$k(X) \simeq \hat{k}_+ = k(+\infty) \quad \text{for} \quad X \geqslant R_1$$

and

$$k(X) \simeq \hat{k}_- = k(-\infty) \quad \text{for} \quad X \leqslant -R_0.$$

For such values of R_0 and R_1, equations (3.3.3 *a*, *b*) become

$$\hat{F}_+(X) \simeq A_+ \exp(\mathrm{i}\sigma^{-2}\tau_+(X)) \quad \text{for} \quad X \geqslant R_1 \tag{3.3.10a}$$

and

$$\hat{F}_-(X) \simeq A_- \exp(\mathrm{i}\sigma^{-2}\tau_-(X)) + B_- \exp(-\mathrm{i}\sigma^{-2}\tau_-(X)) \quad \text{for} \quad X \leqslant -R_0, \tag{3.3.10b}$$

where the constant $A_- = \exp(-\mathrm{i}\sigma^{-2}\hat{k}_- R_0)$ and the constants A_+, B_- are related to the constants $\hat{A}_+$, $\hat{B}_-$. The four constants A_+, B_-, $\bar{C}$ and $\bar{D}$ are specified by the continuity requirements at

$X = -R_0$ and $X = R_1$. Only the values of A_+ and B_- are determined since, for numerical inversion purposes, a more useful equation than (3.3.6) is obtained for the function F in the interval $(-R_0, R_1)$. The values of the constants A_+ and B_- are given in appendix E.

Once an iteration procedure has been developed which evaluates the function F in the interval $(-R_0, R_1)$ and assigns values to the constants A_+ and B_-, then $\hat{F}_\pm(X)$ are calculated from equations (3.3.10 *a*, *b*), $\varphi(X)$ is determined by equation (3.3.9) for each $x \in [-R_0, R_1]$ and new values are assigned to A_+ and B_-.

Iteration form for F in $(-R_0, R_1)$

The integro-differential equations (2.2.9) and (2.2.12) are rearranged using the approximations (3.3.10 *a*, *b*) and expressing the function h as the sum $h_0 + (h - h_0)$, where h_0 is obtained from the typical shape of the particular bottom profile b under consideration. The result, for $X \in (-R_0, R_1)$, is expressible in the form

$$\mathscr{L}F(X) \simeq G(X), \tag{3.3.11}$$

where the linear operator $\mathscr{L}$ and the function G are defined in appendix E. Once a numerical approximation to the inverse L of the operator $\mathscr{L}$ is obtained, and once suitable starting values for $F(X)$, $X \in (-R_0, R_1)$ and the constants A_+ and B_- are given, the equation

$$F(X) \simeq LG(X), \tag{3.3.12}$$

is used iteratively to obtain higher approximations to $F(X)$. It is shown that such an iteration scheme is numerically very useful for a wide range of values of the parameters β and η.

Zero iterate

The initial values for the constants A_+ and B_- are taken as

$$B_- = 0 \quad \text{and} \quad A_+ = \exp(\mathrm{i}\sigma^{-2}\hat{k}_+ R_1), \tag{3.3.13}$$

which is consistent with a uniform depth. The initial function value $F(X)$, $X \in (-R_0, R_1)$ is taken as zero.

Evaluation of the integrals

The integrals that define the function φ for $X \in (-R_0, R_1)$ are of the form

$$\int_{-\infty}^{\infty} H(V)\,\tilde{K}(X, V)\,\mathrm{d}V,$$

where H is a bounded C^2 function (if the solution F is assumed to have this property) and the value $\tilde{K}(X, V)$ of the kernel function is either $\operatorname{cosech}\frac{1}{2}\beta^{-1}(V-X)$ or $\coth\frac{1}{2}\beta^{-1}(V-X) - \operatorname{sgn}(V-X)$. As in the determination of equation (3.3.11), such an integral is approximated by the expression

$$\int_{-R_0}^{R_1} H(V)\,\tilde{K}(X, V)\,\mathrm{d}V + \int_0^{\infty} H_{\mathrm{a}}(V)\,\hat{K}_1(X, V)\,\mathrm{d}V,$$

where $H_{\mathrm{a}}(X)$ is related to the asymptotic estimates of $H(X)$ for $X \geqslant R_1$ and $X \leqslant -R_0$ respectively and the value $\hat{K}_1(X, V)$ of the kernel function is either

$$\operatorname{cosech}(V + \nu(X)) \quad \text{or} \quad \coth\tfrac{1}{2}(V + \nu(X)) - 1,$$

where the function value $\nu(X)$ is real and positive. The Cauchy principal value integrals are evaluated using the technique set out in Kantorovich & Krylov (1958). The infinite integrals remaining are expressed in the form

$$\exp(-\nu(X))\int_0^\infty H_a(V)\,\hat{P}(X,V)\exp(-V)\,\mathrm{d}V,$$

where $\hat{P}$ is a uniformly bounded function of V. The exponential weight function appears explicitly in the integrand and the Gauss–Laguerre quadrature formulae are used to evaluate these integrals (see, for example, Krylov 1962).

The exact value of the integral

$$⨍_{-\infty}^{\infty}\frac{\exp(\mathrm{i}\kappa_0 V)}{\sinh\frac{1}{2}\beta^{-1}(V-X)}\,\mathrm{d}V,$$

which is similar to those defining the function φ, is known by the identity 3.981, 1 of Gradshteyn & Ryzhik (1965). Using the technique described above with the Gauss–Laguerre formula for eight nodes and the trapezium rule with a step size of 0.2 gives acceptable agreement to the exact value for all $X\in(-R_0, R_1)$ and a suitable range of values of the parameter β and wavenumber $d^{-1}\kappa_0$.

To determine the constants A_+ and B_-, the function φ must also be evaluated at the end points $X = -R_0$ and $X = R_1$. The Cauchy principal value integrals that define φ do not exist at their end points, but the integrands have singularities which decay exponentially to zero away from the singularity. As h is a slowly varying function in the vicinity of $X = -R_0$ and $X = R_1$, a reasonable approximation to $\varphi(-R_0)$ and $\varphi(R_1)$ is obtained by replacing the function value $F(X)$ by the expressions that define its local behaviour near $X = -R_0$ and $X = R_1$, that is equations (3.3.10 *a*, *b*). Then, using the dispersion relation (3.1.2) and the identities 3.981,1 and 3.987,2 of Gradshteyn & Ryzhik (1965), it follows that

$$\varphi(-R_0) \simeq \mathrm{i}\sigma^{-2}(A_- - B_-)\,(\hat{k}_- - \pi^2\beta^2 h_-^2\hat{k}_-^{-1}),$$

and

$$\varphi(R_1) \simeq \mathrm{i}\sigma^{-2}A_+(\hat{k}_+ - \pi^2\beta^2 h_+^2\hat{k}_+^{-1}).$$

Matrix representation

The integrals that appear in the definition of the operator $\mathscr{L}$ are also approximated in the manner described above. As the trapezium rule is a second order finite difference process, the derivatives that appear in $\mathscr{L}$ are consistently approximated by the central difference formula. After transferring the quantities $F(-R_0)$ and $F(R_1)$ to the right hand side, this discretization of the continuous operator $\mathscr{L}$ results in a finite set of equations for the function values $F(X_i)$, $X_i\in(-R_0, R_1)$.

The error involved in taking the zeroth iterates as suitable approximations to the function F in the intervals $[R_1, +\infty)$ and $(-\infty, -R_0]$, and of using the constant values $h_\pm$, $\hat{k}_\pm$ for the function values $h(X)$, $k(X)$ in these intervals is essentially of the order of $\exp(-R_1)$, $\exp(-R_0)$. Thus for values of R_0 and R_1 no smaller than 3.5, the percentage errors are no worse than 3 %. Hence for a step size of 0.2 the order (N) of the square matrix to be inverted is no smaller than 35. An approximate inverse of the matrix of coefficients is obtained numerically and a measure of the conditioning of the system of equations is given by the product of the norm of the matrix and the norm of its inverse. The PDP-6 computer at the University of Western Australia – the only machine available at the time the calculations were made – has an inversion package that uses

Gauss–Jordan elimination with total pivoting and calculates this number for the maximum row sum norm. The variation of the conditioning number (cono) for different choices of the constants R_0, R_1, p and q is marked and typical examples are given in table 1 for the function h appropriate to a mound of height 0.4.

An examination of the square matrices of odd dimension for $p = 0$ and $q = 1$ shows them to be dominantly skew symmetric which explains why the conditioning number for these matrices is very large; for all square skew-symmetric matrices of odd dimension are singular.

TABLE 1

β	η	R_0	R_1	N	p	q	cono
0.6	0.52	4.8	4.6	46	0	1	762
0.6	0.3	4.8	4.6	46	0	1	11 230
		4.3	4.3	42	0	1	2 322
		3.8	3.6	36	0	1	703
0.6	0.36	3.8	3.8	37	0	1	$3\,324 \times 10^6$
					0.1	0.9	298
			3.6	36	0	1	1 254
					0.1	0.9	280
					0.4	0.6	241
					0.6	0.4	242

TABLE 2

β	η	R_0	R_1	N	p	q	cono
0.6	0.42	3.8	3.6	36	0	1	25 860
					0.1	0.9	327
					0.2	0.8	396
					0.4	0.6	478
					0.6	0.4	498

The situation is not so critical if the constants R_0 and R_1 are chosen to guarantee that, when $p = 0$ and $q = 1$, the matrix is of even dimension. But in particular cases the dependence of the conditioning number for various choices of the constants p and q is still important, see table 2. When $p = 1$ and $q = 0$ the system of equations appears to be always ill-conditioned; no explanation of this fact has been found.

Experience gained in working many examples using the above inversion package indicates that the constants p, q, R_0 and R_1 should be chosen to achieve a conditioning number no larger than 600. The frequency with which one of the sets of ordered pairs (0, 1), (0.1, 0.9) or (−0.3, 1.3) for (p, q) achieved this result was surprising. Although on some occasions larger values of the conditioning number gave reasonable results, the iteration always converged numerically for conditioning numbers smaller than 600. The number of iterates N_0 required to converge to the function value $F(X_i)$ (to four figure accuracy) at each discrete point in the interval $(-R_0, R_1)$ was always no more than 30 with more iterates required in general for smaller values of the parameter η. The resulting reflexion and transmission coefficients satisfied the energy balance equation, in all cases, to better than 99 %.

A number of functions h have been investigated and in no case was it found impossible to locate suitable values of the constants p and q. However, it is by no means certain that there are not configurations for which the approach considered in this section will not work. If there were to be a bottom geometry for which a trapped wave solution exists, then the integro-differential equations will have eigensolutions. Then there would be no unique function F and so iterative methods would be completely ineffective unless some method could be devised for suppressing the eigensolution. None of the published material on the reflexion of water waves due to depth variations provides any idea as to how this may be achieved.

4. Discussion

4.1. *The function h*

The function h is determined from the conformal mapping $z_1 = Q(\beta z)$ of the strip z-plane to the physical z_1-plane. For general bottom profiles b, it is assumed that conformal mappings $z_0 = Q_0(\beta z)$ are available that approximate to the exact transformation Q. The aim is to select an approximation so that the upper boundaries of the z_1-plane and the z_0-plane are coincident, both 'strips' approach the width ϵ as $\operatorname{Re} z \to \infty$ and the width 1 as $\operatorname{Re} z \to -\infty$, and the lower boundary in the z_0-plane is a 'good' approximation to that in the z_1-plane but lies nowhere above it. The 'strip' in the z_0-plane can then be conformally mapped into the interior of the unit circle in the S-plane by a mapping function $S = \hat{Q}_0(z_0)$. The same mapping function $\hat{Q}_0$ will map the 'strip' in the physical z_1-plane into the interior of a simple closed curve lying nowhere outside the unit circle. There exist many analytical–numerical techniques which determine the conformal mapping of a near unit circle to the unit circle. For example, Kantorovich & Krylov (1958) discuss a number of methods for mapping the interiors of families of bounding curves on to the interior of the unit circle. Moreover, they indicate sufficient conditions for useful convergence of the various techniques. Ideally the basic approximation to the conformal mapping would be chosen to place the lower boundary of the z_0-plane close enough to that of the z_1-plane to ensure the convergence of the chosen technique. The approximating conformal mappings would be obtained from a dictionary of conformal mappings (it would need to be more extensive than that of Kober's (1952)) or else from an appropriate Schwarz–Christoffel transformation.

The latter approach has been used to obtain approximating shapes for the two important families of bottom profiles: I, mounds superimposed on steps; and II, plateaus. The bottom profiles of family I include steps as a limiting case and those of family II include symmetrical mounds (and reefs) as a limiting case. The details of these approximating shapes are tabulated with the w-plane a parallel strip of width π. The significance of the w-plane and its relation to the parallel strip D'' (see § 2.2) becomes apparent later.

I. *Mounds superimposed on steps*:

Conformal mapping, $z_2 = Q_1^{(1)}(w)$: (see figure 1)

$$z_2 = -\mathrm{i}\epsilon + 2\epsilon\pi^{-1}\ln\left(S_3^{\frac{1}{2}} + (S_3-1)^{\frac{1}{2}}\right) - \pi^{-1}\ln\left((1+c)^{\frac{1}{2}}c^{-\frac{1}{2}} + (S_3-1)^{\frac{1}{2}}S_3^{-\frac{1}{2}}\right) + \pi^{-1}\ln\left((1+c)^{\frac{1}{2}}c^{-\frac{1}{2}} - (S_3-1)^{\frac{1}{2}}S_3^{-\frac{1}{2}}\right),$$

where
$$S_3 = -c(1+\exp(w)),$$
$$\delta_0 = 2\epsilon\pi^{-1}\arctan\left((1-\gamma)^{\frac{1}{2}}\gamma^{-\frac{1}{2}}\right) - 2\pi^{-1}\arctan\left((1-\gamma)^{\frac{1}{2}}(1+c)^{-\frac{1}{2}}c^{\frac{1}{2}}\gamma^{-\frac{1}{2}}\right),$$

for each value of $\gamma \in (0, 1)$ and the corresponding value of c given by the expression

$$c = \frac{(1-4\epsilon^2\gamma(1-\gamma))^{\frac{1}{2}} - (1-2\epsilon^2\gamma)}{2(1-\epsilon^2)}.$$

All branch cuts lie in the lower S_3-plane with $S_3^{\frac{1}{2}}$ and $(S_3-1)^{\frac{1}{2}}$ taking real positive values when their arguments are real positive, and each logarithm taking its principal value.

Special case

In the limiting case $\gamma \to 1-$; $c \to \epsilon^2(1-\epsilon^2)^{-1}$ and $\delta_0 \to 0$ which corresponds to a step.

The transformation $Q_1^{(1)}$ maps the lines $\mathrm{Im}\, w = -\alpha$, with $\alpha \in (0, \pi)$, on to smooth curves with asymptotes $\mathrm{Im}\, z_2 = -\epsilon\alpha\pi^{-1}$ and $\mathrm{Im}\, z_2 = -\alpha\pi^{-1}$ at $\mathrm{Re}\, z_2 = \pm\infty$ respectively since $Q_1^{(1)}$ has zero distortion at the ends. Consider the linear transformations $z_1 = \pi\alpha^{-1}z_2$ and $w = \alpha\pi^{-1}z$ composed with the above transformation $Q_1^{(1)}$ for each value of $\alpha \in (0, \pi)$. Then the conformal mapping of the strip z-plane of width π to the z_1-plane with the lower boundary a smooth mound superimposed on a smooth step approaching the depths ϵd and d as $x_1 \to \pm\infty$ is obtained. This family of bottom shapes is parametrized by α and γ.

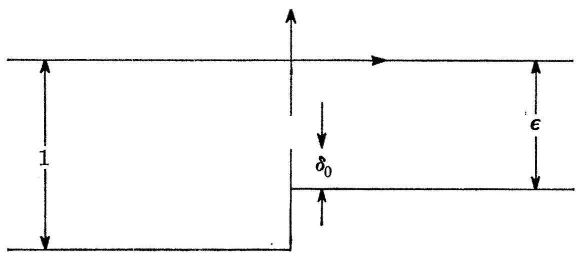

FIGURE 1. z_2-plane.

Let the parameter $\beta = \alpha\pi^{-1} \in (0, 1]$. Then β is a measure of the scale length of the obstacle; the value of $\beta = 1$ corresponding to a bottom profile with a discontinuous change in depth and the limiting case of β approaching zero corresponding to the 'transition width' becoming infinite.

A sketch of the bottom profiles corresponding to the three values of the parameter $\beta = 1$, 0.9 and 0.6 is given in figure 2. Also in the special case when the value of the parameter $\gamma = 1$, a sketch of the bottom profiles corresponding to these three values of β is given in figure 3. The function h can be computed and is given by the equation

$$h(X) = \left.\frac{\mathrm{d}z_1}{\mathrm{d}z}\right|_{y=0} = \frac{\epsilon}{\pi}\frac{(c+\gamma)+c\exp(X)}{(c+1)+c\exp(X)}\left[\frac{(1+c)+c\exp(X)}{c(1+\exp(X))}\right]^{\frac{1}{2}}.$$

II. *Plateaus*:

Conformal mapping, $z_2 = Q_1^{(2)}(w)$: (see figure 4)

$$\frac{\mathrm{d}z_2}{\mathrm{d}w} = \frac{1}{\pi}\left[\frac{\mu^2-1}{\mu^2-l^2}\right]^{\frac{1}{2}}\left[\frac{S_4^2-l^2}{S_4^2-1}\right]^{\frac{1}{2}},$$

where $S_4 = \mu\coth(\tfrac{1}{2}w)$,

$$H_0 = 1-\Lambda_0(\arcsin[(\mu^2-1)^{\frac{1}{2}}(\mu^2-l^2)^{-\frac{1}{2}}]\backslash\arcsin(1-l^2)^{\frac{1}{2}}), \quad (4.1)$$

and

$$L_0 = 2\pi^{-1}\mu^{-1}l^2(\mu^2-1)^{\frac{1}{2}}(\mu^2-l^2)^{-\frac{1}{2}}K(\arcsin l) - 2\pi^{-1}K(\arcsin l)\,Z^*(\arcsin(\mu^{-1})\backslash\arcsin l), \quad (4.2)$$

for each value of $\mu \in (1, \infty)$ and $l \in (0, 1)$. Here K is the complete elliptic integral of the first kind, and Λ_0 and Z^* are the tabulated Heuman's lambda function and Jacobian zeta function respectively. All branch cuts lie in the lower S_4-plane with $(S_4^2-l^2)^{\frac{1}{2}}$ and $(S_4^2-1)^{\frac{1}{2}}$ taking real positive values when their arguments are real positive.

Special case

In the limiting case $l \to 0+$; $L_0 \to 0$ and $\mu \to \operatorname{cosec}(\frac{1}{2}\pi H_0)$ which corresponds to a vertical barrier.

The transformation $Q_1^{(2)}$ maps the lines $\operatorname{Im} w = -\alpha$, with $\alpha \in (0, \pi)$, on to smooth curves with asymptotes $\operatorname{Im} z_2 = -\alpha\pi^{-1}$ at $\operatorname{Re} z_2 = \pm\infty$ since $Q_1^{(2)}$ has zero distortion at the ends. Consider the linear transformations $z_1 = \pi\alpha^{-1}z_2$ and $w = \alpha\pi^{-1}z$ composed with the above transformation $Q_1^{(2)}$

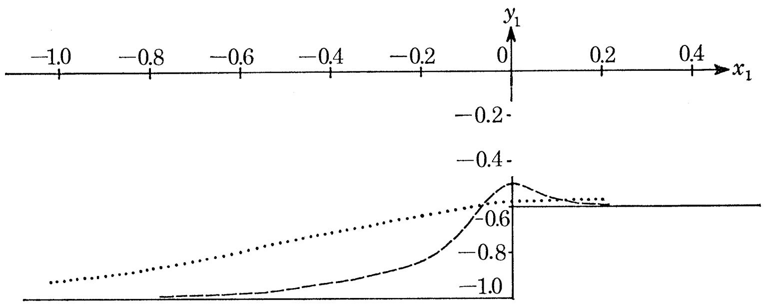

FIGURE 2. Mounds approaching different depths as $x_1 \to \pm\infty$. $\delta_0 = 0.14$, $\gamma = 0.4$, $c = 0.08$. ——, $\beta = 1$ (vertical barrier on a discontinuous step); ----, $\beta = 0.9$; ......, $\beta = 0.6$.

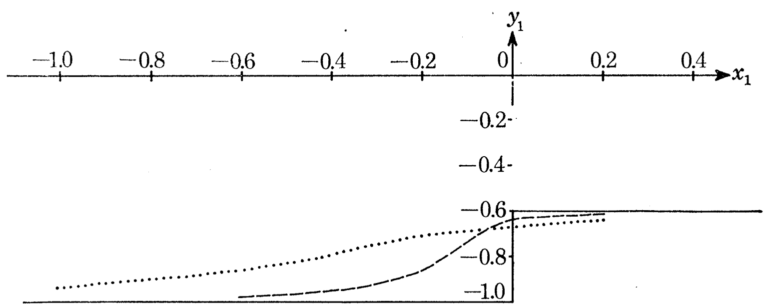

FIGURE 3. Steps of height 0.4. ——, $\beta = 1$ (discontinuous step); ----, $\beta = 0.9$; ......, $\beta = 0.6$.

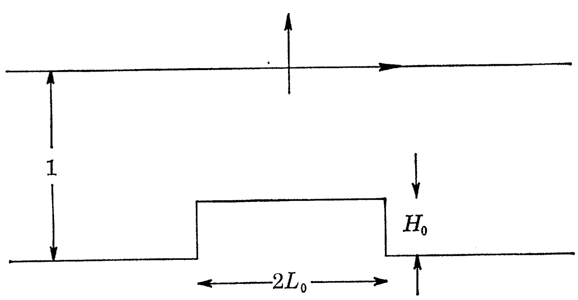

FIGURE 4. z_2-plane.

for each value of $\alpha \in (0, \pi)$. Then the conformal mapping of the strip z-plane of width π to the z_1-plane with the lower boundary a smooth elongated mound approaching the depth d as $x_1 \to \pm\infty$ is obtained. This family of bottom shapes is parametrized by α, μ and l. Each pair of values of the parameters μ and l defines a rectangular block of height H_0 and half-width L_0. Two numerical examples have been obtained by using the tables of Abramowitz & Stegun (1964), namely,

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

(i) for $\arcsin(1-l^2)^{\frac{1}{2}} = 30°$ and $\arcsin[(\mu^2-1)^{\frac{1}{2}}(\mu^2-l^2)^{-\frac{1}{2}}] = 40°$, there corresponds a block height H_0 of approximately 0.398 units and a width $2L_0$ of approximately 0.67 units, and

(ii) for $\arcsin(1-l^2)^{\frac{1}{2}} = 68°$ and $\arcsin[(\mu^2-1)^{\frac{1}{2}}(\mu^2-l^2)^{-\frac{1}{2}}] = 50°$ there corresponds a block height H_0 of approximately 0.400 units and a width $2L_0$ of approximately 0.08 units.

The inverse problem may be treated as follows. For each value of $l \in (0, 1)$ choose the value of $\mu \in (1, \infty)$ that gives the required height H_0. Then choose that pair of values (l, μ) that yields the required half-width L_0. No details of the plateau shapes have been pursued save to notice that, holding l and μ fixed, as β is decreased the height of the resulting plateau is decreased and its

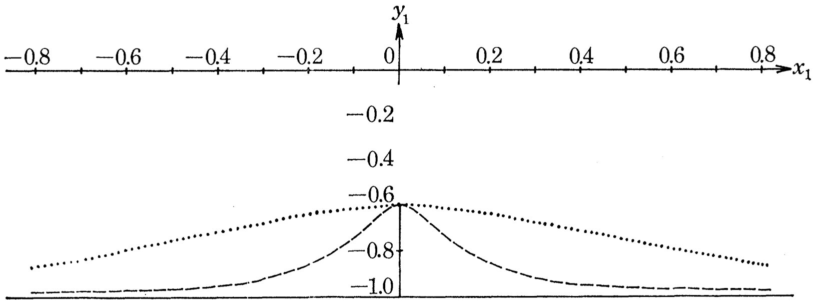

FIGURE 5. Mounds of height 0.4. ——, $\beta = 1$ (vertical barrier); ----, $\beta = 0.9$; ......, $\beta = 0.6$.

transition width is increased. In the special case when $l = 0$, the barrier height ρ $(= H_0)$ that yields a mound of height $\hat{\rho}$ is related to the parameter β by the expression

$$\cos(\tfrac{1}{2}\pi\rho) = \frac{\sin(\frac{1}{2}\pi\beta(1-\hat{\rho}))}{\sin(\frac{1}{2}\pi\beta)}.$$

A sketch of the bottom profiles corresponding to a mound of height 0.4 for the three values of $\beta = 1$, 0.9 and 0.6 is given in figure (5). The function h can be computed and is given by the equation

$$h(X) = \left.\frac{\mathrm{d}z_1}{\mathrm{d}z}\right|_{y=0} = \frac{1}{\pi}\left[\frac{\mu^2-1}{\mu^2-l^2}\right]^{\frac{1}{2}}\left[\frac{\mu^2-l^2\tanh^2(\frac{1}{2}X)}{\mu^2-\tanh^2(\frac{1}{2}X)}\right]^{\frac{1}{2}}.$$

An alternative conformal mapping of the z-plane to a region with a smooth step-like lower boundary has been given by Roseau (1952). In the above notation, his transformation is defined by the formula

$$z_1 = Q(\beta z) = \pi^{-1}z + (\pi\beta)^{-1}(\epsilon-1)\ln(1+\exp(\beta z)),$$

where the value of the parameter β is restricted to the open interval (0, 1). The corresponding function h is given by the equation

$$h(X) = \pi^{-1}(1+\epsilon\exp(X))(1+\exp(X))^{-1}.$$

It is suggested in Kajiura (1963) that the effective width $\hat{l}$ of the transition zone is given by the expression

$$\hat{l} = |\mathrm{Re}\,Q(\pi-\mathrm{i}\pi\beta)| + |\mathrm{Re}\,Q(-\pi-\mathrm{i}\pi\beta)|.$$

Thus if the function values $Q(\pm\pi-\mathrm{i}\pi\beta)$ are estimated by neglecting the term $\exp(-\pi)$ in comparison with unity, and the value of the parameter ϵ is not too small, then the stated approximation to $\hat{l}$ is obtained, namely $\hat{l} \simeq \beta^{-1}(1+\epsilon)$.

The procedure of generating bottom profiles using the Schwarz–Christoffel transformation suggests that a great variety of shapes can be obtained. Using the transformation for plateaus as a typical example the technique is, for each fixed value of the parameter $\beta \in (0, 1]$, and for all values of the parameters $\mu \in (1, \infty)$ and $l \in (0, 1)$, to compute the function value $Q(X - i\pi\beta)$. The resulting obstacles $y_1 = -b(x_1)$ are then identified parametrically in terms of the equations $x_1 = \mathrm{Re}\, Q(X - i\pi\beta)$ and $y_1 = \mathrm{Im}\, Q(X - i\pi\beta)$. In particular, if $\beta = 1$ each rectangular block has the dimensions H_0 and $2L_0$ as given by equations (4.1) and (4.2).

The results of this section are consistent with the form in which the assumptions on the behaviour of the function h have been made. In general this is likely to be the behaviour of the function h.

4.2. *Results*

The methods developed in § 3 are now used to extend the range of problems for which numerical solutions are available. Specifically, the reflexion coefficient is obtained as a function of both the parameters η and σ^{-1} for the bottom profiles considered in § 4.1.

Description of the method

For a chosen function h and an assumed value of the parameter β, the matrix iteration scheme, if it converges, will determine the reflexion and transmission coefficients for each value of the parameter η. But for fixed values of the parameters R_0, R_1 and N (which means the function value $h(X)$ may be stored as an array), the matrix iteration scheme can be expected to experience difficulties once a significant number of wavelengths appear in the transition zone. That is, for values of the parameter σ^{-1} greater than a certain value, the matrix iteration scheme will no longer be particularly useful.

For the functions h considered, it was observed that the matrix iteration scheme and the asymptotic formula (3.1.15) gave significantly different results for the values of the parameters $\beta = 0.6$ and $\sigma^{-1} > 1.4$. The value of $\sigma^{-1} = 1.4$ was therefore taken as the limit of usefulness of the matrix iteration scheme. For larger values of σ^{-1}, the reflexion coefficient was determined using the formula (3.1.15). Possibly this value of $\sigma^{-1} = 1.4$ may be increased by adopting any one, or all of the following variations in the numerical process:

(i) reducing the step size,

(ii) increasing the number of iterates used to represent the wave in the two intervals $(-\infty, -R_0]$ and $[R_1, \infty)$,

(iii) increasing the value of the parameters R_0 and/or R_1.

The results were obtained by using the PDP-6 computer at the University of Western Australia. This machine was being superseded at the time and in its run down condition prevented a discussion of the items (i), (ii) and (iii). Any modification of the numerical procedure, however, must be considered in the light of the alternative descriptions that are available, that is the asymptotic formulae of §§ 3.1 and 3.2. The method used in a particular instance to be decided by the machine time involved in computing the value of the reflexion coefficient.

For the bottom profiles considered here, it was not found necessary to use the asymptotic results of § 3.2. Nevertheless, when the value of the parameter σ^{-1} is small, the results given in § 3.2 define the slope of the curve of $|\hat{R}|$ against σ^{-1}.

The complete curves were obtained by joining the curves for values of the parameter σ^{-1} greater than and less than 1.4 through the point $\sigma^{-1} = 1.4$. The only suspicious results were those in a small neighbourhood of the value of $\sigma^{-1} = 1.4$; and here $|\hat{R}|$ was only 3 % of the incident wave amplitude. It thus appeared that the errors involved in the numerical computations, for these values of the parameter η, were of the same order as the value of $|\hat{R}|$. It is the author's opinion that the present difficulty in the vicinity of this value of $\sigma^{-1} = 1.4$ could be overcome by employing the first of the variations suggested above, that is by reducing the step size used in the numerical computations.

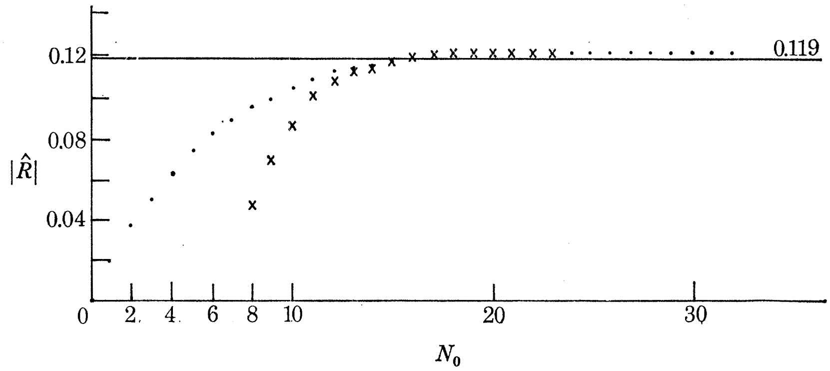

FIGURE 6. N_0, number of iterates; $|\hat{R}|^2$, reflexion coefficient. Roseau's bottom profile for $\epsilon = 0.6$, $\eta = 0.06$, $\beta = 0.95$; 'exact' value for $|\hat{R}|$ is 0.1194. ×, results for an incorrect seventh iterate.

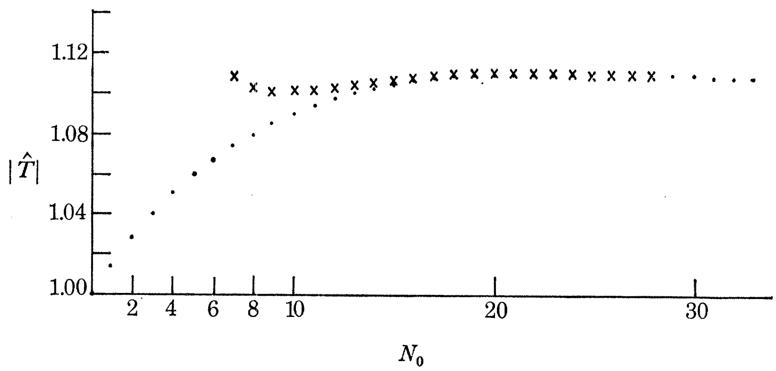

FIGURE 7. N_0, number of iterates; $|\hat{T}|^2$, transmission coefficient. Roseau's bottom profile for $\epsilon = 0.6$, $\eta = 0.06$, $\beta = 0.95$. ×, results for an incorrect seventh iterate.

Assessment of the matrix iteration scheme

The matrix iteration scheme was applied to the Roseau bottom profile and to a vertical barrier of height 0.5 so that a direct comparison could be made with the exact formula of Roseau (1952) and with the approximate results of Mei & Black (1969) obtained using a variational approach.

Comparison with Roseau

For the values of the parameters $\epsilon = 0.6$, $\beta = 0.95$ and $\eta = 0.06$, Roseau's (1952) exact formula yields, for the value of the square root of the reflexion coefficient, $|\hat{R}| = 0.1194$. A sketch of the values of $|\hat{R}|$ and the square root of the transmission coefficient $|\hat{T}|$, obtained for the successive iterates of the matrix iteration scheme of § 3.3, are given in figures 6 and 7 respectively. The operator $\mathscr{L}$ of equation (3.3.11) was approximated by a square matrix of order 46. In one of the computer runs an incorrect seventh iterate was introduced which did not affect the numerically determined limit suggesting the iteration scheme is usefully stable. In either case, the first 20 or so iterates have converged to values for the reflexion and transmission coefficients. The higher iterates shown in figure 6 marginally improve the approximation. Thirty iterates yield a value of the reflexion coefficient with an error of 3 %; this is of the same order as the errors involved in developing the numerical method. While in practice this error is perhaps too large, it should be emphasized that the approximation is obtained using the zero approximation to the wave in the intervals $(-\infty, -R_0]$ and $[R_1, \infty)$ and a discrete representation of the linear operator $\mathscr{L}$ based on a not very small step size. Thus the modifications of the method outlined in (i), (ii), and (iii) above should reduce the error to meet any practical limits that might be imposed.

Comparison with Mei & Black (1969)

The comparison is necessarily qualitative since no error estimates are available for Mei & Black's (1969) variational method of estimating the reflexion coefficient. The values of the parameters used were $R_0 = 4.8$, $R_1 = 4.6$ and $N = 46$. The two curves are given in figure 8. The

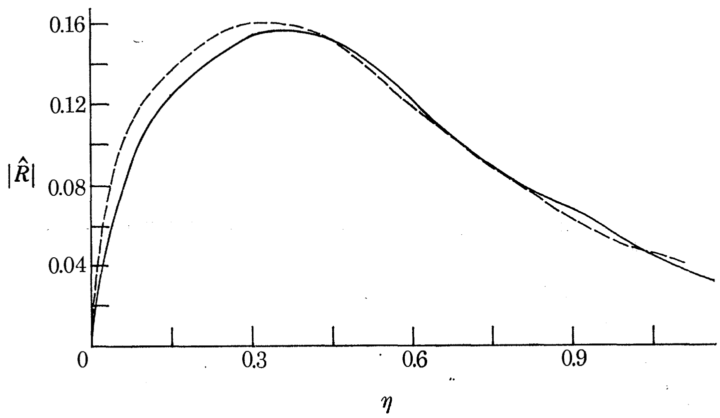

FIGURE 8. Vertical barrier of height 0.5. ——, Method of this paper; ----, Mei & Black's (1969) result.

maximum absolute difference between them is 0.017 at the value of the parameter $\eta = 0.15$. This value corresponds to a wavelength of approximately eight times the depth d and a value of $|\hat{R}|$ approximately equal to 0.13. For small values of η, the curve obtained by the method of this paper is on the same side of Mei & Black's (1969) curve as the long wave approximation of Ogilvie (1960). Inasmuch as there is some error involved in transferring Mei & Black's (1969) curve to figure 8, the two methods produce essentially the same curve. However, the method of Mei & Black (1969) must be preferred (computationally) to that of this paper on those bottom profiles where their method is immediately applicable. In particular for a rectangular block whose width is of the order of, or larger than the depth d, the parameters R_0 and R_1 would need to be chosen quite large to ensure the value of $\pi h(X)$ is close to unity in the intervals $(-\infty, -R_0]$ and $[R_1, \infty)$. This means, for reasonably small step sizes, a large system of algebraic equations must be solved. The number of computations involved in using Mei & Black's (1969) approximate solution technique would be much less. The present method has been used here essentially to provide confirmation of the results obtained.

Bottom profiles of specific interest

The existence of obstacles for which the reflexion coefficient is zero for certain wavelengths has been established by Newman (1965). In the long wave theory, bottom profiles with this filtering property are also known (see for example, Kajiura 1963).

The bottom profiles of specific interest here are derived from mounds superimposed on steps and plateaus (including the special cases $\gamma = 1$ and $l = 0$) using the three values of the parameter

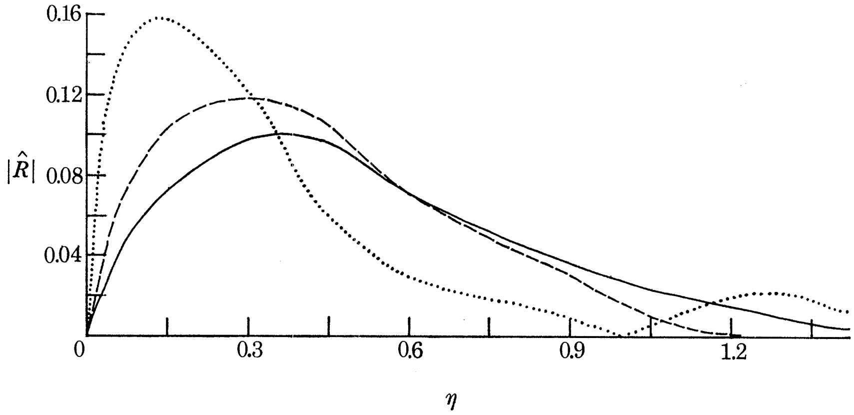

FIGURE 9. Mounds of height 0.4 approaching the same depth.
——, $\beta = 1.0$; ----, $\beta = 0.9$; ......, $\beta = 0.6$.

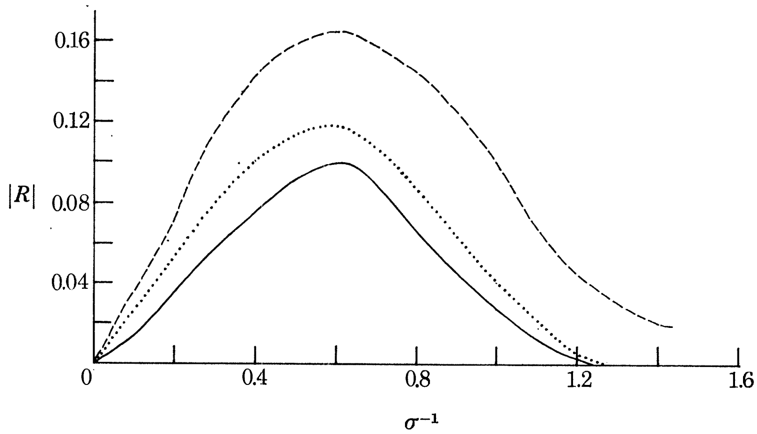

FIGURE 10. Mounds of height 0.4 approaching the same depth.
——, $\beta = 1.0$; ......, $\beta = 0.9$; ----, $\beta = 0.6$.

$\beta = 1$, 0.9 and 0.6. When $\sigma^{-1} > 1.4$, the reflexion coefficient is evaluated using the formula (3.1.15). Thus the value of η for which total transmission occurs can be determined exactly. For $\sigma^{-1} < 1.4$ the values of η for which the reflexion coefficient is zero cannot be determined as precisely.

Mounds

The mounds considered are shown in figure 5. In figures 9 and 10 the composite curves of $|\hat{R}|$ against η, and σ^{-1} are exhibited. The maximum reflexion for the three mounds occurs when the value of the parameter σ^{-1} is approximately 0.6. Thus when the value of the parameter β is 1,

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A

the maximum reflexion occurs when the wavelength is approximately five times the depth d. When the value of β is 0.9, the maximum reflexion occurs when the wavelength is approximately $5.5d$, and for the value of $\beta = 0.6$, when the wavelength is approximately $9d$. As the wavelength and transition width increase, in such a way that the value of σ^{-1} remains constant, the reflexion coefficient increases as the value of β decreases. This observation is a simple consequence of a larger wavelength being required to maintain the constant value of σ^{-1}. Furthermore, for fixed values of the parameter $\eta > 0.6$, that is for wavelengths greater than $3d$, the reflexion coefficient decreases as the transition width increases.

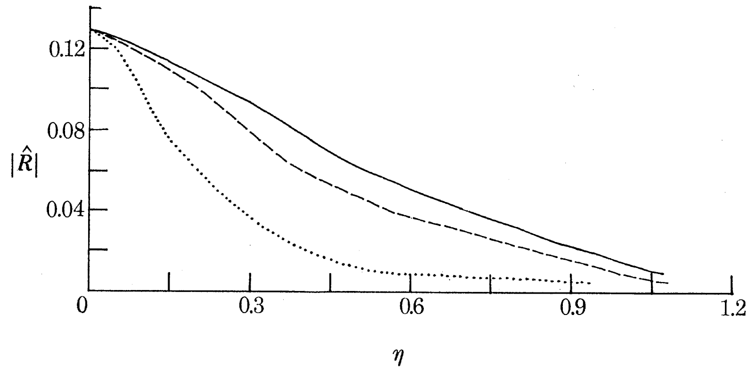

FIGURE 11. Steps of height 0.4. ——, $\beta = 1$; ----, $\beta = 0.9$; ......, $\beta = 0.6$.

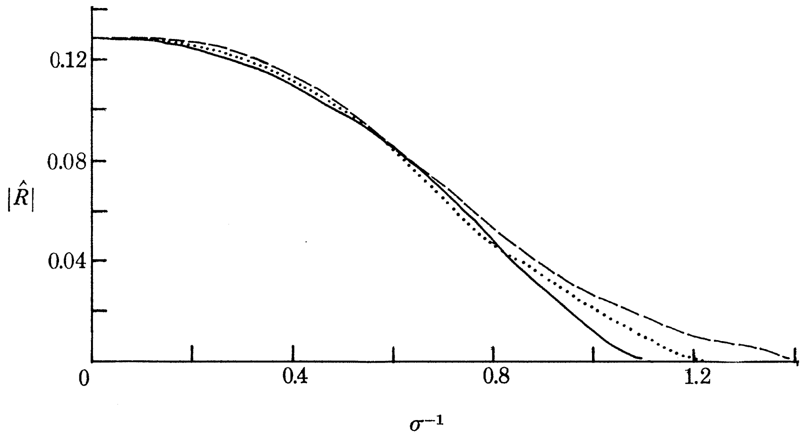

FIGURE 12. Steps of height 0.4. ——, $\beta = 1$; ......, $\beta = 0.9$; ----, $\beta = 0.6$.

For the bottom profile corresponding to the value of the parameter $\beta = 0.6$, the transition width is approximately three times the depth d. When the value of the parameter η is 1, which corresponds to a wavelength of approximately $2d$, this bottom profile allows total transmission.

Steps

The steps considered are shown in figure 3. In figures 11 and 12 the resulting composite curves of $|\hat{R}|$ against η, and σ^{-1} are exhibited. The value of $|\hat{R}|$, in the limit of long wavelengths, corresponds to the long wave formula of Lamb. For all finite fixed values of the wavelength, the reflexion coefficient decreases as the transition width increases. Also, for fixed values of the parameter $\sigma^{-1} > 0.8$, the reflexion coefficient increases as the transition width increases. This result agrees with that for the mounds and is again a consequence of the wavelength increasing to maintain the constant value of σ^{-1} for increasing transition widths.

Plateaus

Details of the plateaus considered have not been pursued save to notice that the height of the determining rectangular block is approximately 0.4 times the depth d and its width lies between $0.08d$ and $0.67d$. In figures 13 and 14 the composite curves of $|\hat{R}|$ against η, and σ^{-1} are exhibited.

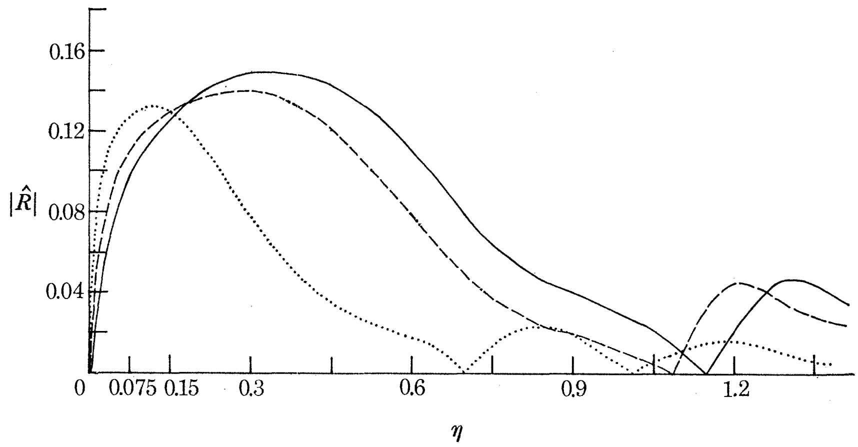

FIGURE 13. Plateaus. $\arcsin(1-l^2)^{\frac{1}{2}} = 50°$, $\arcsin((\mu^2-1)^{\frac{1}{2}}(\mu^2-l^2)^{-\frac{1}{2}}) = 45°$.
——, $\beta = 1$; – – – –, $\beta = 0.9$; ……, $\beta = 0.6$.

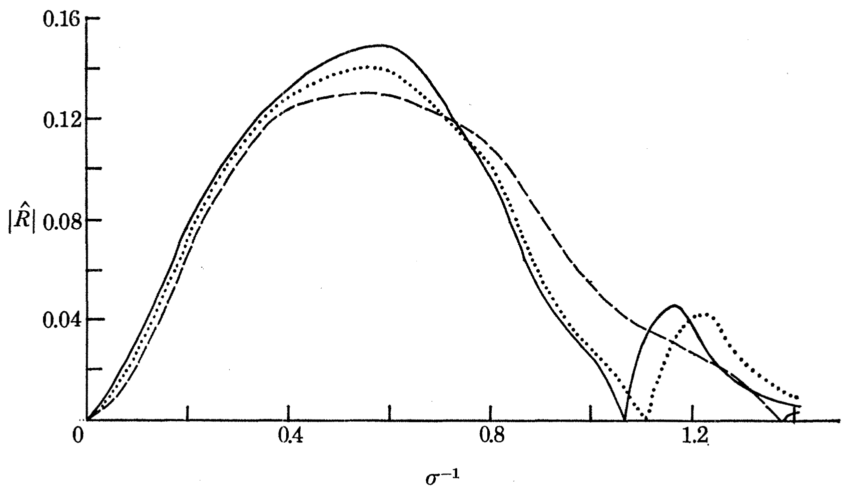

FIGURE 14. Plateaus. $\arcsin(1-l^2)^{\frac{1}{2}} = 50°$, $\arcsin((\mu^2-1)^{\frac{1}{2}}(\mu^2-l^2)^{-\frac{1}{2}}) = 45°$.
——, $\beta = 1$; ……, $\beta = 0.9$; – – – –, $\beta = 0.6$.

The maximum reflexion for the three bottom profiles occurs when the value of the parameter σ^{-1} is approximately 0.6, as for the mounds. For fixed values of $\sigma^{-1} \in (0, 0.7)$, less reflexion occurs from the wider transition zones. This is presumably because the effective height of the plateaus is then smaller. For fixed values of $\sigma^{-1} \in (0.7, 1.1)$, more reflexion occurs from the wider transition zones, in agreement with the two previous shapes. Also, for fixed values of the parameter $\eta \in (0.15, 1.05)$, the reflexion coefficient decreases as the transition width increases.

When the value of the parameter β is 1, the method of the paper yields a value of the wavelength of approximately twice the depth d for which the reflexion coefficient is zero. This result is consistent with those of Mei & Black (1969) since, for a fixed block height, as the width decreases, the first zero of the reflexion coefficient occurs at decreasing values of the wavelength.

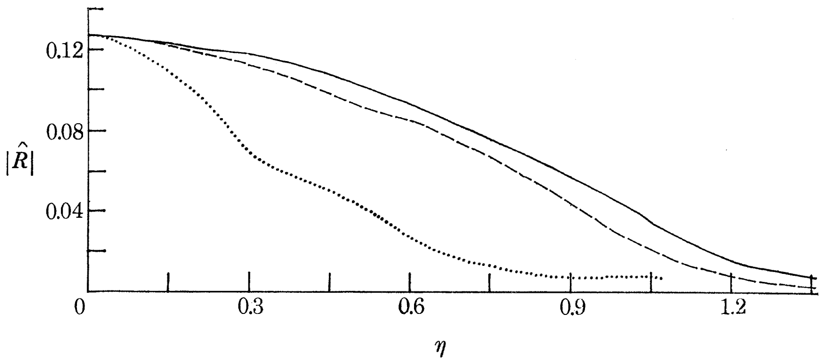

FIGURE 15. Mounds approaching depths 0.6 and 1 as $x_1 \to \pm\infty$.
——, $\beta = 1$; – – – –, $\beta = 0.9$; ......, $\beta = 0.6$.

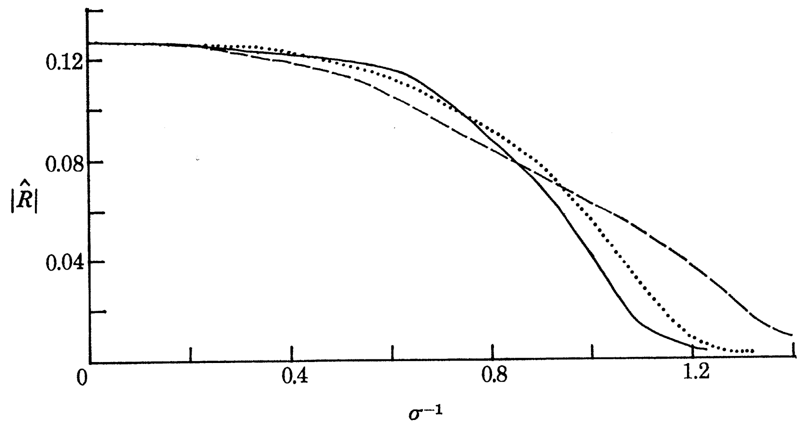

FIGURE 16. Mounds approaching depths 0.6 and 1 as $x_1 \to \pm\infty$.
——, $\beta = 1$; ......, $\beta = 0.9$; – – – –, $\beta = 0.6$.

Moreover the method of this paper also indicates that the smooth bottom profiles, corresponding to the values of the parameter $\beta = 0.9$ and $\beta = 0.6$, have the same filtering property. When the value of β is 0.9, the smallest value of the parameter η is approximately 1.08, corresponding to a wavelength of approximately twice the depth d, and when the value of β is 0.6 the smallest value of η is approximately 0.7, corresponding to a wavelength of approximately $3d$.

Mounds superimposed on steps

The bottom profiles considered for these shapes are shown in figure 2. In figures 15 and 16 composite curves of $|\hat{R}|$ against η, and σ^{-1} are exhibited. The value of $|\hat{R}|$, in the limit of long wavelengths, corresponds to the long wave result of Lamb for a step of height 0.4, since in this limit the reflexion coefficient for a vertical barrier in a finite constant depth of fluid is zero.

For all fixed finite wavelengths, less reflexion occurs from wider transition zones. However, the reflexion coefficients for the values of the parameters $\beta = 1$ and $\beta = 0.9$ agree for the values of

$\eta \in (0, 0.15)$ and stay close to the value 0.126 over a significant range of values of η. It appears reasonable to expect (basing this conjecture on the results obtained for the vertical barrier in a fluid of constant depth) that for higher barriers than the one considered, there may exist a range of values of η in which the reflexion coefficient will be larger than that corresponding to the limiting case of values of η approaching zero.

For fixed values of the parameter $\sigma^{-1} \geqslant 1$, greater reflexion occurs from wider transition zones. This follows as before, since to maintain the constant value of σ^{-1} the wavelength and transition width must increase together.

I am grateful to Professor J. J. Mahony for his supervision of my doctoral thesis of which this paper is the major part and to the members of the Mathematics Department of the University of Western Australia for many helpful discussions. I thank the administrators of the Commonwealth Postgraduate Award scheme for financial assistance in the course of my research. Finally, I thank the referees for their comments which improved the presentation.

Appendix A

The function P

The value of the left hand side of equation (3.1.5) is $2P(X)$ by virtue of the condition (3.1.6). Thus an alternative expression for the function P, in terms of the underived functions T and R, is immediately known.

A simplification of the latter expression is obtained by first introducing the change of variable

$$w = \theta(X, V) = \tau(X+V) - \tau(X) = \int_X^{X+V} k(U)\,\mathrm{d}U, \tag{A 1}$$

where (for each fixed value of X) θ is a monotone function of V and then extracting the 'principal part' of each resulting integral. Let the value of V which satisfies equation (A 1) be given by the formula $V = \hat{\theta}(X, w)$. Then the term 'principal part', in relation to the expression

$$\tfrac{1}{2}(\pi\beta)^{-2} \exp(\mathrm{i}\sigma^{-2}\tau(X)) ⨍_{-\infty}^{\infty} \frac{T(X+\hat{\theta}(X,w)) \exp(\mathrm{i}\sigma^{-2}w)}{k(X+\hat{\theta}(X,w)) \sinh \frac{1}{2}\beta^{-1}\hat{\theta}(X,w)}\,\mathrm{d}w,$$

which is typical of the integral terms appearing in the right hand side of equation (3.1.5), refers to the expression

$$\tfrac{1}{2}(\pi\beta)^{-2} \frac{T(X)\exp(\mathrm{i}\sigma^{-2}\tau(X))}{k(X)} ⨍_{-\infty}^{\infty} \frac{\exp(\mathrm{i}\sigma^{-2}w)}{\sinh\frac{1}{2}(\beta k(X))^{-1}w}\,\mathrm{d}w.$$

The sum of all such 'principal parts' cancels with the term

$$-\mathrm{i}\sigma^{-2}k(X)\,T(X)\exp(\mathrm{i}\sigma^{-2}\tau(X)) + \mathrm{i}\sigma^{-2}k(X)\,R(X)\exp(-\mathrm{i}\sigma^{-2}\tau(X))$$

by virtue of the identity

$$\tfrac{1}{2}\sigma^{-2}h(X)\,(k(X))^{-1} ⨍_{-\infty}^{\infty} (\coth\tfrac{1}{2}(\beta k(X))^{-1}w - \operatorname{sgn} w)\exp(\mathrm{i}\sigma^{-2}w)\,\mathrm{d}w$$
$$+\frac{1}{2\pi^2\beta^2k(X)} ⨍_{-\infty}^{\infty} \frac{\exp(\mathrm{i}\sigma^{-2}w)}{\sinh\frac{1}{2}(\beta k(X))^{-1}w}\,\mathrm{d}w - \mathrm{i}\sigma^{-2}k(X) = 0,$$

obtained using the formulae 3.981, 1 and 3.987, 2 of Gradshteyn & Ryzhik (1965) and the dispersion relation (3.1.2).

Hence the value of the function P is expressible as

$$P(X) = \frac{\exp(\mathrm{i}\sigma^{-2}\tau(X))}{4\sigma^2} \times \int_{-\infty}^{\infty}\left[\left(\frac{hT}{k}\right)(X+\hat{\theta}(X,w))\left(\coth\frac{\hat{\theta}(X,w)}{2\beta}-\operatorname{sgn} w\right)-\left(\frac{hT}{k}\right)(X)\left(\coth\frac{w}{2\beta k(X)}-\operatorname{sgn} w\right)\right] \times \exp(\mathrm{i}\sigma^{-2}w)\,\mathrm{d}w + \frac{\exp(-\mathrm{i}\sigma^{-2}\tau(X))}{4\sigma^2}$$
$$\times \int_{-\infty}^{\infty}\left[\left(\frac{hR}{k}\right)(X+\hat{\theta}(X,w))\left(\coth\frac{\hat{\theta}(X,w)}{2\beta}-\operatorname{sgn} w\right)-\left(\frac{hR}{k}\right)(X)\left(\coth\frac{w}{2\beta k(X)}-\operatorname{sgn} w\right)\right] \times \exp(-\mathrm{i}\sigma^{-2}w)\,\mathrm{d}w + \frac{\exp(\mathrm{i}\sigma^{-2}\tau(X))}{4\pi^2\beta^2}$$
$$\times \int_{-\infty}^{\infty}\left[\left(\frac{T}{k}\right)(X+\hat{\theta}(X,w))\frac{1}{\sinh\frac{1}{2}\beta^{-1}\hat{\theta}(X,w)}-\left(\frac{T}{k}\right)(X)\frac{1}{\sinh\frac{1}{2}(\beta k(X))^{-1}w}\right] \times \exp(\mathrm{i}\sigma^{-2}w)\,\mathrm{d}w + \frac{\exp(-\mathrm{i}\sigma^{-2}\tau(X))}{4\pi^2\beta^2}$$
$$\times \int_{-\infty}^{\infty}\left[\left(\frac{R}{k}\right)(X+\hat{\theta}(X,w))\frac{1}{\sinh\frac{1}{2}\beta^{-1}\hat{\theta}(X,w)}-\left(\frac{R}{k}\right)(X)\frac{1}{\sinh\frac{1}{2}(\beta k(X))^{-1}w}\right] \times \exp(-\mathrm{i}\sigma^{2}w)\,\mathrm{d}w. \quad \text{(A 2)}$$

The operator matrix $\mathscr{M}$

Define the linear operators $\mathscr{T}$, $\mathscr{R}$, $\mathscr{L}_T$ and $\mathscr{L}_R$ as follows

$$\mathscr{T}\tilde{g}(X) = \int_{-\infty}^{X}\tilde{g}(V)\exp(-\mathrm{i}\sigma^{-2}\tau(V))\,\mathrm{d}V,$$

$$\mathscr{R}\tilde{g}(X) = -\int_{X}^{\infty}\tilde{g}(V)\exp(\mathrm{i}\sigma^{-2}\tau(V))\,\mathrm{d}V,$$

and

$$\mathscr{L}_{\{{T \atop R}\}}\tilde{g}(X) = \frac{\exp(\pm\mathrm{i}\sigma^{-2}\tau(X))}{4\sigma^2} \times \int_{-\infty}^{\infty}\left[\left(\frac{h\tilde{g}}{k}\right)(X+\hat{\theta}(X,w))\left(\coth\frac{\hat{\theta}(X,w)}{2\beta}-\operatorname{sgn} w\right)-\left(\frac{h\tilde{g}}{k}\right)(X)\left(\coth\frac{w}{2\beta k(X)}-\operatorname{sgn} w\right)\right] \times \exp(\pm\mathrm{i}\sigma^{-2}w)\,\mathrm{d}w + \frac{\exp(\pm\mathrm{i}\sigma^{-2}\tau(X))}{4\pi^2\beta^2}$$
$$\times \int_{-\infty}^{\infty}\left[\left(\frac{\tilde{g}}{k}\right)(X+\hat{\theta}(X,w))\frac{1}{\sinh\frac{1}{2}\beta^{-1}\hat{\theta}(X,w)}-\left(\frac{\tilde{g}}{k}\right)(X)\frac{1}{\sinh\frac{1}{2}(\beta k(X))^{-1}w}\right] \times \exp(\pm\mathrm{i}\sigma^{-2}w)\,\mathrm{d}w.$$

Then $\mathscr{M}$ is the operator matrix

$$\begin{bmatrix}\mathscr{T}\circ\mathscr{L}_T & \mathscr{T}\circ\mathscr{L}_R \\ \mathscr{R}\circ\mathscr{L}_T & \mathscr{R}\circ\mathscr{L}_R\end{bmatrix}.$$

The term $\mathscr{M}\mathscr{F}$ appearing in equation (3.1.8) is to be interpreted in the obvious way.

Appendix B

Lemma 1 is now proved.

Lemma 1. Suppose the functions $T, R \in \mathscr{B}$. Then there exists a $\sigma_0 > 0$ such that

$$|P(X)| \leqslant K\sigma^2\|(T,R)\|\exp(-|X|) \quad (\forall X \in \mathbb{R}),$$

whenever the parameter $\sigma \in (0,\sigma_0)$, where K is a constant independent of σ.

Proof. The lemma is proved by using equation (A 2) and the assumed behaviour as $X \to \pm\infty$ of the analytic function h. Firstly, it follows immediately from the definition that $h \in \mathscr{B} \cap C^2$. Also, as the problem of interest is the effect a given bottom profile has on a range of prescribed incident waves, h is independent of the parameter σ. Thus, by the alternative form of the dispersion relation (3.1.3), the function $k \in \mathscr{B} \cap C^2$ and the two functions k and $X \mapsto k'(X)\exp(|X|)$ are bounded independently of σ for sufficiently small values of σ. The partial derivatives

$$(X, w) \mapsto \hat{\theta}_w(X, w) \quad \text{and} \quad (X, w) \mapsto \hat{\theta}_{ww}(X, w)$$

therefore exist and for sufficiently small values of σ are bounded independently of σ.

To estimate the order of the function P, the standard method of approximating Fourier integrals with large wavenumbers is used. The first two integrals in equation (A 2) can be written in the form

$$\int_0^\infty (E(X, w)\cos\sigma^{-2}w + \mathrm{i}O(X, w)\sin\sigma^{-2}w)\,\mathrm{d}w,$$

where E and O are even and odd functions of w respectively, neither one of which involves the generalized function sgn. Thus sgn does not contribute to the estimates of these integrals and for convenience, the estimation of the function value $P(X)$ is discussed in terms of the doubly infinite integrals.

For fixed values of X, the value of each integrand appearing in equation (A 2) decays exponentially to zero as $w \to \pm\infty$. Thus, in the limiting case of $\sigma \to 0$, the value of each integral in equation (A 2) is determined by the small w behaviour of the respective terms in [ ]. The last two integrals have the estimate $\|(T, R)\|\, S_1(X)\, o(\sigma^2)$, where the error term $o(\sigma^2)$ is uniform in X, the function S_1 is bounded independently of σ for sufficiently small values of σ and the product $S_1(X)\exp(|X|)$ approaches finite limits as $X \to \pm\infty$. This result follows immediately after performing an integration by parts, using the mean value theorem and the exponential decaying properties of the derivatives h', h'', T' and R', and then applying the Reimann–Lebesgue lemma. Similarly, the first two integrals have the estimate

$$\frac{\mathrm{i}}{4}\int_{-\infty}^{\infty}\left[\left(\frac{hT'}{k^2}\right)(X+\hat{\theta}(X,w))\exp(\mathrm{i}\sigma^{-2}(\tau(X)+w)) - \left(\frac{hR'}{k^2}\right)(X+\hat{\theta}(X,w))\exp(-\mathrm{i}\sigma^{-2}(\tau(X)+w))\right] \times\left[\coth\frac{\hat{\theta}(X,w)}{2\beta} - \operatorname{sgn} w\right]\mathrm{d}w + \|(T,R)\|\,S_2(X)\,o(\sigma^2), \quad \text{as} \quad \sigma \to 0. \quad (\text{B } 1)$$

The error term $o(\sigma^2)$ is uniform in X and the function S_2 has the same properties as the function S_1. This last approximation is a consequence of the fact that the functions h, k and coth are C^2 functions and the function $(X, w) \mapsto \hat{\theta}(X, w)$ possesses two partial derivatives with respect to w, which are bounded independently of σ for sufficiently small values of σ; accordingly one further integration by parts can be performed on these integrals. But the integral in the estimate (B 1) is identically zero, by assumption (3.1.6).

The lemma is an immediate consequence of the above estimates involving the functions S_1 and S_2.

Appendix C

The functions $G_\pm$

The values of the left hand sides of equations (3.2.5 *a*, *b*) are $\sigma^{-2}G_\pm(X)$ and the derivative F' involves only the underived functions $T_\pm$ and $R_\pm$, by virtue of the conditions (3.2.6 *a*, *b*). Thus alternative expressions for the functions $G_\pm$, in terms of the underived functions $T_\pm$, $R_\pm$ and F, are immediately known.

A simplification of these expressions is obtained by extracting the 'principal parts' of the Cauchy principal value integrals, as in appendix A. In this case the sum of the 'principal parts' cancels with the terms $(h(\pm\infty)-k_\pm^2)\,F(X)$ by virtue of the identities

$$\tfrac{1}{2}\mathrm{i}\sigma^{-1}k_\pm h(\pm\infty)\,⨍_{-\infty}^{\infty}\exp(\mathrm{i}\sigma^{-1}k_\pm V)\,(\coth\tfrac{1}{2}\beta^{-1}V-\operatorname{sgn}V)\,\mathrm{d}V+k_\pm^2-h(\pm\infty)=0,$$

obtained by using the dispersion relations (3.2.3 a, b) and the identity 3.987, 2 of Gradshteyn & Ryzhik (1965).

Hence after extensive, but straightforward calculations, the values of the functions $G_\pm$ are expressible as

$$\begin{aligned}G_+(X)=&\frac{1}{2}\int_0^\infty((h'(V)+\mathrm{i}\sigma^{-1}k_+h(V))\,T_+(V)-(h'(X)+\mathrm{i}\sigma^{-1}k_+h(X))\,T_+(X))\\&\qquad\times\exp(\mathrm{i}\sigma^{-1}k_+V)\,\bar{K}(V,X)\,\mathrm{d}V\\&+\frac{1}{2}\int_0^\infty((h'(V)-\mathrm{i}\sigma^{-1}k_+h(V))\,R_+(V)-(h'(X)-\mathrm{i}\sigma^{-1}k_+h(X))\,R_+(X))\\&\qquad\times\exp(-\mathrm{i}\sigma^{-1}k_+V)\,\bar{K}(V,X)\,\mathrm{d}V\\&+\frac{1}{2}\int_{-\infty}^0[h'(V)\,[T_-(V)\exp(\mathrm{i}\sigma^{-1}k_-V)+R_-(V)\exp(-\mathrm{i}\sigma^{-1}k_-V)]\\&-h'(X)\,[T_+(X)\exp(\mathrm{i}\sigma^{-1}k_+V)+R_+(X)\exp(-\mathrm{i}\sigma^{-1}k_+V)]]\,\bar{K}(V,X)\,\mathrm{d}V\\&+\frac{1}{2}\int_{-\infty}^0[\mathrm{i}\sigma^{-1}k_-h(V)\,[T_-(V)\exp(\mathrm{i}\sigma^{-1}k_-V)-R_-(V)\exp(-\mathrm{i}\sigma^{-1}k_-V)]\\&-\mathrm{i}\sigma^{-1}k_+h(X)\,[T_+(X)\exp(\mathrm{i}\sigma^{-1}k_+V)-R_+(X)\exp(-\mathrm{i}\sigma^{-1}k_+V)]]\,\bar{K}(V,X)\,\mathrm{d}V\\&-\frac{\pi^2\beta^2\sigma^{-1}k_+}{\pi\beta\tanh\pi\beta\sigma^{-1}k_+}(T_+(X)\exp(\mathrm{i}\sigma^{-1}k_+X)+R_+(X)\exp(-\mathrm{i}\sigma^{-1}k_+X))\,(h(X)-h(+\infty))\\&+\mathrm{i}\frac{\pi\beta\sigma^{-1}k_+-\tanh\pi\beta\sigma^{-1}k_+}{\sigma^{-1}k_+\tanh\pi\beta\sigma^{-1}k_+}(T_+(X)\exp(\mathrm{i}\sigma^{-1}k_+X)-R_+(X)\exp(-\mathrm{i}\sigma^{-1}k_+X))\,h'(X)\end{aligned}\tag{C 1}$$

and

$$\begin{aligned}G_-(X)=&\frac{1}{2}\int_{-\infty}^0((h'(V)+\mathrm{i}\sigma^{-1}k_-h(V))\,T_-(V)-(h'(X)+\mathrm{i}\sigma^{-1}k_-h(X))\,T_-(X))\\&\qquad\times\exp(\mathrm{i}\sigma^{-1}k_-V)\,\bar{K}(V,X)\,\mathrm{d}V\\&+\frac{1}{2}\int_{-\infty}^0((h'(V)-\mathrm{i}\sigma^{-1}k_-h(V))\,R_-(V)-(h'(X)-\mathrm{i}\sigma^{-1}k_-h(X))\,R_-(X))\\&\qquad\times\exp(-\mathrm{i}\sigma^{-1}k_-V)\,\bar{K}(V,X)\,\mathrm{d}V\\&+\frac{1}{2}\int_0^\infty[h'(V)\,[T_+(V)\exp(\mathrm{i}\sigma^{-1}k_+V)+R_+(V)\exp(-\mathrm{i}\sigma^{-1}k_+V)]\\&-h'(X)\,[T_-(X)\exp(\mathrm{i}\sigma^{-1}k_-V)+R_-(X)\exp(-\mathrm{i}\sigma^{-1}k_-V)]]\,\bar{K}(V,X)\,\mathrm{d}V\\&+\frac{1}{2}\int_0^\infty[\mathrm{i}\sigma^{-1}k_+h(V)\,[T_+(V)\exp(\mathrm{i}\sigma^{-1}k_+V)-R_+(V)\exp(-\mathrm{i}\sigma^{-1}k_+V)]\\&-\mathrm{i}\sigma^{-1}k_-h(X)\,[T_-(X)\exp(\mathrm{i}\sigma^{-1}k_-V)-R_-(X)\exp(-\mathrm{i}\sigma^{-1}k_-V)]]\,\bar{K}(V,X)\,\mathrm{d}V\\&-\frac{\pi^2\beta^2\sigma^{-1}k_-}{\pi\beta\tanh\pi\beta\sigma^{-1}k_-}(T_-(X)\exp(\mathrm{i}\sigma^{-1}k_-X)+R_-(X)\exp(-\mathrm{i}\sigma^{-1}k_-X))\,(h(X)-h(-\infty))\\&+\mathrm{i}\frac{\pi\beta\sigma^{-1}k_--\tanh\pi\beta\sigma^{-1}k_-}{\sigma^{-1}k_-\tanh\pi\beta\sigma^{-1}k_-}(T_-(X)\exp(\mathrm{i}\sigma^{-1}k_-X)-R_-(X)\exp(-\mathrm{i}\sigma^{-1}k_-X))\,h'(X),\end{aligned}\tag{C 2}$$

where the function $\bar{K}$ is defined by the equation

$$\begin{aligned}\bar{K}(X, Y) &= \coth \tfrac{1}{2}\beta^{-1}(X-Y) - \operatorname{sgn}(X-Y) \\ &= \operatorname{sgn}(X-Y)\,\frac{2\exp(-\beta^{-1}|X-Y|)}{1-\exp(-\beta^{-1}|X-Y|)}.\end{aligned}$$

The operator matrix $\mathscr{M}$

Define the linear operators $\mathscr{J}_1 - \mathscr{J}_4$, $\mathscr{R}_{1,2}$ and $\mathscr{L}_1^{\pm} - \mathscr{L}_4^{\pm}$ as follows

$$\mathscr{J}_1(\tilde{g}) = \int_0^\infty \tilde{g}(V)\exp(\mathrm{i}\sigma^{-1}k_+V)\,\mathrm{d}V,$$

$$\mathscr{J}_2(\tilde{g}) = \int_{-\infty}^0 \tilde{g}(V)\exp(-\mathrm{i}\sigma^{-1}k_-V)\,\mathrm{d}V,$$

$$\mathscr{J}_3\tilde{g}(X) = \int_0^X \tilde{g}(V)\exp(-\mathrm{i}\sigma^{-1}k_+V)\,\mathrm{d}V,$$

$$\mathscr{J}_4\tilde{g}(X) = \int_{-\infty}^X \tilde{g}(V)\exp(-\mathrm{i}\sigma^{-1}k_-V)\,\mathrm{d}V,$$

$$\mathscr{R}_1\tilde{g}(X) = \int_X^\infty \tilde{g}(V)\exp(\mathrm{i}\sigma^{-1}k_+V)\,\mathrm{d}V,$$

$$\mathscr{R}_2\tilde{g}(X) = \int_X^0 \tilde{g}(V)\exp(\mathrm{i}\sigma^{-1}k_-V)\,\mathrm{d}V,$$

$$\begin{aligned}\mathscr{L}^{+}_{\{1\}\{3\}}\tilde{g}(X) = {}& \frac{1}{2}\int_0^\infty ((h'(V) \pm \mathrm{i}\sigma^{-1}k_+h(V))\,\tilde{g}(V) - (h'(X) \pm \mathrm{i}\sigma^{-1}k_+h(X))\,\tilde{g}(X)) \\ & \qquad\qquad \times \exp(\pm\mathrm{i}\sigma^{-1}k_+V)\,\bar{K}(V,X)\,\mathrm{d}V \\ & -\tfrac{1}{2}(h'(X) \pm \mathrm{i}\sigma^{-1}k_+h(X))\,\tilde{g}(X)\int_{-\infty}^0 \exp(\pm\mathrm{i}\sigma^{-1}k_+V)\,\bar{K}(V,X)\,\mathrm{d}V \\ & -\frac{\pi^2\beta^2\sigma^{-1}k_+}{\pi\beta\tanh\pi\beta\sigma^{-1}k_+}\,\tilde{g}(X)\exp(\pm\mathrm{i}\sigma^{-1}k_+X)\,(h(X)-h(+\infty)) \\ & \pm\mathrm{i}\,\frac{\pi\beta\sigma^{-1}k_+ - \tanh\pi\beta\sigma^{-1}k_+}{\sigma^{-1}k_+\tanh\pi\beta\sigma^{-1}k_+}\,\tilde{g}(X)\exp(\pm\mathrm{i}\sigma^{-1}k_+X)\,h'(X),\end{aligned}$$

$$\mathscr{L}^{+}_{\{2\}\{4\}}\tilde{g}(X) = \frac{1}{2}\int_{-\infty}^0 (h'(V) \pm \mathrm{i}\sigma^{-1}k_-h(V))\,\tilde{g}(V)\exp(\pm\mathrm{i}\sigma^{-1}k_-V)\,\bar{K}(V,X)\,\mathrm{d}V,$$

$$\begin{aligned}\mathscr{L}^{-}_{\{1\}\{3\}}\tilde{g}(X) = {}& \frac{1}{2}\int_{-\infty}^0 ((h'(V) \pm \mathrm{i}\sigma^{-1}k_-h(V))\,\tilde{g}(V) - (h'(X) \pm \mathrm{i}\sigma^{-1}k_-h(X))\,\tilde{g}(X)) \\ & \qquad\qquad \times \exp(\pm\mathrm{i}\sigma^{-1}k_-V)\,\bar{K}(V,X)\,\mathrm{d}V \\ & -\tfrac{1}{2}(h'(X) \pm \mathrm{i}\sigma^{-1}k_-h(X))\,\tilde{g}(X)\int_0^\infty \exp(\pm\mathrm{i}\sigma^{-1}k_-V)\bar{K}(V,X)\,\mathrm{d}V \\ & -\frac{\pi^2\beta^2\sigma^{-1}k_-}{\pi\beta\tanh\pi\beta\sigma^{-1}k_-}\,\tilde{g}(X)\exp(\pm\mathrm{i}\sigma^{-1}k_-X)\,(h(X)-h(-\infty)) \\ & \pm\mathrm{i}\,\frac{\pi\beta\sigma^{-1}k_- - \tanh\pi\beta\sigma^{-1}k_-}{\sigma^{-1}k_-\tanh\pi\beta\sigma^{-1}k_-}\,\tilde{g}(X)\exp(\pm\mathrm{i}\sigma^{-1}k_-X)\,h'(X)\end{aligned}$$

and

$$\mathscr{L}^{-}_{\{2\}\{4\}}\tilde{g}(X) = \frac{1}{2}\int_0^\infty (h'(V) \pm \mathrm{i}\sigma^{-1}k_+h(V))\,\tilde{g}(V)\exp(\pm\mathrm{i}\sigma^{-1}k_+V)\,\bar{K}(V,X)\,\mathrm{d}V.$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

To simplify the notation, introduce the linear operators $\hat{\mathscr{L}}_1^{\pm}-\hat{\mathscr{L}}_4^{\pm}$ as follows:

$$\hat{\mathscr{L}}_1^{+} = -\tfrac{1}{2}\mathrm{i}k_+^{-1}\mathscr{J}_3\circ\mathscr{L}_1^{+}-\tfrac{1}{2}\mathrm{i}k_+^{-1}\frac{k_+-k_-}{k_++k_-}\mathscr{J}_1\circ\mathscr{L}_1^{+}-\frac{\mathrm{i}}{k_++k_-}\mathscr{J}_2\circ\mathscr{L}_2^{-},$$

$$\hat{\mathscr{L}}_2^{+} = -\tfrac{1}{2}\mathrm{i}k_+^{-1}\mathscr{J}_3\circ\mathscr{L}_2^{+}-\tfrac{1}{2}\mathrm{i}k_+^{-1}\frac{k_+-k_-}{k_++k_-}\mathscr{J}_1\circ\mathscr{L}_2^{+}-\frac{\mathrm{i}}{k_++k_-}\mathscr{J}_2\circ\mathscr{L}_1^{-},$$

$$\hat{\mathscr{L}}_3^{+} = -\tfrac{1}{2}\mathrm{i}k_+^{-1}\mathscr{J}_3\circ\mathscr{L}_3^{+}-\tfrac{1}{2}\mathrm{i}k_+^{-1}\frac{k_+-k_-}{k_++k_-}\mathscr{J}_1\circ\mathscr{L}_3^{+}-\frac{\mathrm{i}}{k_++k_-}\mathscr{J}_2\circ\mathscr{L}_4^{-},$$

$$\hat{\mathscr{L}}_4^{+} = -\tfrac{1}{2}\mathrm{i}k_+^{-1}\mathscr{J}_3\circ\mathscr{L}_4^{+}-\tfrac{1}{2}\mathrm{i}k_+^{-1}\frac{k_+-k_-}{k_++k_-}\mathscr{J}_1\circ\mathscr{L}_4^{+}-\frac{\mathrm{i}}{k_++k_-}\mathscr{J}_2\circ\mathscr{L}_3^{-},$$

$$\hat{\mathscr{L}}_1^{-} = -\tfrac{1}{2}\mathrm{i}k_-^{-1}\mathscr{R}_2\circ\mathscr{L}_2^{-}+\tfrac{1}{2}\mathrm{i}k_-^{-1}\frac{k_+-k_-}{k_++k_-}\mathscr{J}_2\circ\mathscr{L}_2^{-}-\frac{\mathrm{i}}{k_++k_-}\mathscr{J}_1\circ\mathscr{L}_1^{+},$$

$$\hat{\mathscr{L}}_2^{-} = -\tfrac{1}{2}\mathrm{i}k_-^{-1}\mathscr{R}_2\circ\mathscr{L}_1^{-}+\tfrac{1}{2}\mathrm{i}k_-^{-1}\frac{k_+-k_-}{k_++k_-}\mathscr{J}_2\circ\mathscr{L}_1^{-}-\frac{\mathrm{i}}{k_++k_-}\mathscr{J}_1\circ\mathscr{L}_2^{+},$$

$$\hat{\mathscr{L}}_3^{-} = -\tfrac{1}{2}\mathrm{i}k_-^{-1}\mathscr{R}_2\circ\mathscr{L}_4^{-}+\tfrac{1}{2}\mathrm{i}k_-^{-1}\frac{k_+-k_-}{k_++k_-}\mathscr{J}_2\circ\mathscr{L}_4^{-}-\frac{\mathrm{i}}{k_++k_-}\mathscr{J}_1\circ\mathscr{L}_3^{+}$$

and

$$\hat{\mathscr{L}}_4^{-} = -\tfrac{1}{2}\mathrm{i}k_-^{-1}\mathscr{R}_2\circ\mathscr{L}_3^{-}+\tfrac{1}{2}\mathrm{i}k_-^{-1}\frac{k_+-k_-}{k_++k_-}\mathscr{J}_2\circ\mathscr{L}_3^{-}-\frac{\mathrm{i}}{k_++k_-}\mathscr{J}_1\circ\mathscr{L}_4^{+}.$$

Then $\mathscr{M}$ is the operator matrix

$$\mathscr{M} = \begin{bmatrix} \hat{\mathscr{L}}_1^{+} & \hat{\mathscr{L}}_2^{+} & \hat{\mathscr{L}}_3^{+} & \hat{\mathscr{L}}_4^{+} \\ -\tfrac{1}{2}\mathrm{i}k_-^{-1}\mathscr{J}_4\circ\mathscr{L}_2^{-} & -\tfrac{1}{2}\mathrm{i}k_-^{-1}\mathscr{J}_4\circ\mathscr{L}_1^{-} & -\tfrac{1}{2}\mathrm{i}k_-^{-1}\mathscr{J}_4\circ\mathscr{L}_4^{-} & -\tfrac{1}{2}\mathrm{i}k_-^{-1}\mathscr{J}_4\circ\mathscr{L}_3^{-} \\ -\tfrac{1}{2}\mathrm{i}k_+^{-1}\mathscr{R}_1\circ\mathscr{L}_1^{+} & -\tfrac{1}{2}\mathrm{i}k_+^{-1}\mathscr{R}_1\circ\mathscr{L}_2^{+} & -\tfrac{1}{2}\mathrm{i}k_+^{-1}\mathscr{R}_1\circ\mathscr{L}_3^{+} & -\tfrac{1}{2}\mathrm{i}k_+^{-1}\mathscr{R}_1\circ\mathscr{L}_4^{+} \\ \hat{\mathscr{L}}_1^{-} & \hat{\mathscr{L}}_2^{-} & \hat{\mathscr{L}}_3^{-} & \hat{\mathscr{L}}_4^{-} \end{bmatrix}.$$

The term $\mathscr{M}\mathscr{F}$ appearing in equation (3.2.11) is also to be interpreted in the obvious way.

Appendix D

Lemma 3 is now proved.

Lemma 3. Suppose the functions $T_{\pm}$, $R_{\pm}\in\mathscr{B}^{\pm}$. Then there exists a $\boldsymbol{\sigma}_0>0$ such that

$$|G_{\pm}(X)| \leqslant \boldsymbol{K}\,\|(T_+,T_-,R_+,R_-)\|\exp\left(-\tfrac{1}{2}|X|\right) \quad (\forall X \gtrless 0),$$

whenever the parameter $\sigma>\boldsymbol{\sigma}_0$, where $\boldsymbol{K}$ is a constant independent of σ.

Proof. The proof uses equation (C 1, C 2) and the assumed behaviour as $X\to\pm\infty$ of the analytic function h. Firstly, it follows immediately from the definition (restricting its domain to the intervals $[0,+\infty)$ and $(-\infty,0]$ respectively) that $h\in\mathscr{B}^{\pm}\cap C^2$ and the product $h''(X)\exp\left(\tfrac{1}{2}|X|\right)$ tends to finite limits as $X\to\pm\infty$. Also, as explained in appendix B, the function h is independent of the parameter σ. Thus by the alternative forms of the dispersion relations (3.2.4 *a*, *b*), the wavenumbers $k_{\pm}$ are bounded independently of σ for sufficiently large values of σ.

The last two terms in equations (C 1) and (C 2) define functions $\Phi_{\pm}^{(1)}$ with domains $[0,+\infty)$ and $(-\infty,0]$ respectively. They have the bounds

$$|\Phi_{\pm}^{(1)}(X)| \leqslant \text{const}\,\|(T_+,T_-,R_+,R_-)\|\exp\left(-\tfrac{1}{2}|X|\right),$$

where the constant is independent of σ for sufficiently large values of σ.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The third and fourth terms in equations (C 1) and (C 2) define functions $\Phi_{\pm}^{(2)}$. They have the bounds

$$|\Phi_{+}^{(2)}(X)| \leqslant \text{const} \|(T_+, T_-, R_+, R_-)\| \int_{-\infty}^{0} \bar{K}(V, X)\, \mathrm{d}V$$
$$\leqslant \text{const} \|(T_+, T_-, R_+, R_-)\| \exp(-\tfrac{1}{2}X)$$

and

$$|\Phi_{-}^{(2)}(X)| \leqslant \text{const} \|(T_+, T_-, R_+, R_-)\| \int_{0}^{\infty} \bar{K}(V, X)\, \mathrm{d}V$$
$$\leqslant \text{const} \|(T_+, T_-, R_+, R_-)\| \exp(\tfrac{1}{2}X),$$

where the constants are independent of σ for sufficiently large values of σ.

Finally, the first two terms in equations (C 1) and (C 2) define functions $\Phi_{\pm}^{(3)}$. The function value $\Phi_{+}^{(3)}(X)$ is expressible in the form

$$\Phi_{+}^{(3)}(X) = \frac{1}{2}\int_{0}^{\infty} [(H_1(V)\exp(-\tfrac{1}{2}V) - H_1(X)\exp(-\tfrac{1}{2}X))\exp(\mathrm{i}\sigma^{-1}k_+ V)$$
$$+ (H_2(V)\exp(-\tfrac{1}{2}V) - H_2(X)\exp(-\tfrac{1}{2}X))\exp(-\mathrm{i}\sigma^{-1}k_+ V)]\, \bar{K}(V, X)\, \mathrm{d}V,$$

where the functions $H_{1,2}$ are defined by the equations

$$H_1(X) = ((h'T_+)(X) + \mathrm{i}\sigma^{-1}k_+((hT_+)(X) - (hT_+)(+\infty)))\exp(\tfrac{1}{2}X)$$

and

$$H_2(X) = ((h'R_+)(X) - \mathrm{i}\sigma^{-1}k_+((hR_+)(X) - (hR_+)(+\infty)))\exp(\tfrac{1}{2}X).$$

The functions $H_{1,2}$ and $H'_{1,2}$ have the bounds

$$|H_{1,2}(X)| \leqslant \text{const} \|(T_+, T_-, R_+, R_-)\|$$

and

$$|H'_{1,2}(X)| \leqslant \text{const} \|(T_+, T_-, R_+, R_-)\|,$$

where the constants are independent of σ for sufficiently large values of σ.

A suitable estimate is now derived for the function value $\Phi_{+}^{(3)}(X)$. It is readily shown that

$$\tfrac{1}{2}\exp(\tfrac{1}{2}X)\int_{0}^{\infty} (H_{1,2}(V)\exp(-\tfrac{1}{2}V) - H_{1,2}(X)\exp(-\tfrac{1}{2}X))\exp(\pm\mathrm{i}\sigma^{-1}k_+ V)\, \bar{K}(V, X)\, \mathrm{d}V$$
$$= -\int_{0}^{X} \frac{(H_{1,2}(X-V) - H_{1,2}(X))\exp(\tfrac{1}{2}V)}{\exp(\beta^{-1}V) - 1}\exp(\pm\mathrm{i}\sigma^{-1}k_+(X-V))\, \mathrm{d}V$$
$$-H_{1,2}(X)\int_{0}^{X} \frac{\exp(\tfrac{1}{2}V) - 1}{\exp(\beta^{-1}V) - 1}\exp(\pm\mathrm{i}\sigma^{-1}k_+(X-V))\, \mathrm{d}V$$
$$+\int_{0}^{\infty} \frac{(H_{1,2}(X+V) - H_{1,2}(X))\exp(-\tfrac{1}{2}V)}{\exp(\beta^{-1}V) - 1}\exp(\pm\mathrm{i}\sigma^{-1}k_+(X-V))\, \mathrm{d}V$$
$$-H_{1,2}(X)\int_{0}^{\infty} \frac{\exp(\tfrac{1}{2}V) - 1}{\exp(\beta^{-1}V) - 1}\exp(-\tfrac{1}{2}V)\exp(\pm\mathrm{i}\sigma^{-1}k_+(X-V))\, \mathrm{d}V.$$

Since $\beta \in (0, 1]$, the second and fourth of the above integrals are each bounded by constants which are independent of σ for sufficiently large values of σ. To estimate the first and third integrals the mean value theorem is used to replace the terms $H_{1,2}(X \pm V) - H_{1,2}(X)$ by $\pm VH'_{1,2}(\xi_{1,2})$ for suitable values of $\xi_{1,2}$. Then applying the bounds for the functions $H_{1,2}$ and $H'_{1,2}$ yields the estimate

$$|\Phi_{+}^{(3)}(X)| \leqslant \text{const} \|(T_+, T_-, R_+, R_-)\| \exp(-\tfrac{1}{2}X),$$

where the constant is independent of σ for sufficiently large values of σ. Similarly the function $\Phi_-^{(3)}$ has the estimate

$$|\Phi_-^{(3)}(X)| \leqslant \text{const} \,\|(T_+, T_-, R_+, R_-)\| \exp\left(\tfrac{1}{2}X\right),$$

where the constant is independent of σ for sufficiently large values of σ.

The lemma follows immediately from the above estimates of the functions $\Phi_\pm^{(j)}, j = 1, 2, 3$.

Appendix E

The constants A_+ and B_-

The constants A_+ and B_- are given by the expressions

$$\begin{aligned}\Delta A_+ = {} & 4\overline{w}\sigma^{-2}h_+\hat{k}_-A_- + 2\mathrm{i}\overline{w}h_+\varphi(-R_0) \\ & -2\sigma^{-2}\hat{k}_-\varphi(R_1)\sin\tau_0(R_1) - 2\mathrm{i}\overline{w}h_-\varphi(R_1)\cos\tau_0(R_1) \\ & -2\mathrm{i}\overline{w}^2h_+h_-\int_{-R_0}^{R_1}\varphi(V)\sin\tau_0(V)\,\mathrm{d}V + 2\overline{w}\sigma^{-2}h_+\hat{k}_-\int_{-R_0}^{R_1}\varphi(V)\cos\tau_0(V)\,\mathrm{d}V,\end{aligned} \tag{E 1}$$

and

$$\begin{aligned}\Delta B_- = {} & 2\mathrm{i}(\overline{w}^2h_+h_- - \sigma^{-4}\hat{k}_+\hat{k}_-)\sin\tau_0(R_1)\,A_- \\ & -2\overline{w}\sigma^{-2}(h_-\hat{k}_+ - h_+\hat{k}_-)\cos\tau_0(R_1)A_- - 2\mathrm{i}\overline{w}h_-\varphi(R_1) \\ & +2\sigma^{-2}\hat{k}_+\varphi(-R_0)\sin\tau_0(R_1) + 2\mathrm{i}\overline{w}h_+\varphi(-R_0)\cos\tau_0(R_1) \\ & -2\overline{w}\sigma^{-2}h_-\hat{k}_+\int_{-R_0}^{R_1}\varphi(V)\cos\left(\tau_0(R_1)-\tau_0(V)\right)\mathrm{d}V \\ & +2\mathrm{i}\overline{w}^2h_+h_-\int_{-R_0}^{R_1}\varphi(V)\sin\left(\tau_0(R_1)-\tau_0(V)\right)\mathrm{d}V,\end{aligned} \tag{E 2}$$

where

$$\Delta = -2\mathrm{i}(\overline{w}^2h_+h_- - \sigma^{-4}\hat{k}_+\hat{k}_-)\sin\tau_0(R_1) + 2\overline{w}\sigma^{-2}(h_+\hat{k}_- + h_-\hat{k}_+)\cos\tau_0(R_1). \tag{E 3}$$

The linear operator $\mathscr{L}$ and the function G

The linear operator $\mathscr{L}$ appearing in equation (3.3.11) is defined by the equation

$$\begin{aligned}\mathscr{L}\tilde{g}(X) = {} & p\left[2\pi^2\eta h_0(X)\,\tilde{g}(X) + \fint_{-R_0}^{R_1}\frac{\tilde{g}'(V)}{\sinh\frac{1}{2}\beta^{-1}(V-X)}\,\mathrm{d}V\right] \\ & +q\left[\tilde{g}'(X) - \tfrac{1}{2}\sigma^{-2}\fint_{-R_0}^{R_1}h_0(V)\,\tilde{g}(V)\left(\coth\tfrac{1}{2}\beta^{-1}(V-X) - \operatorname{sgn}(V-X)\right)\mathrm{d}V\right. \\ & \left. -\tfrac{1}{2}(\pi\beta)^{-2}\fint_{-R_0}^{R_1}\frac{\tilde{g}(V)}{\sinh\frac{1}{2}\beta^{-1}(V-X)}\,\mathrm{d}V\right],\end{aligned}$$

where p and q are arbitrary real numbers satisfying the equation $p+q = 1$. The function G is defined as the linear combination $pG_1 + qG_2$ where the functions G_1 and G_2 are specified by the equations

$$\begin{aligned}G_1(X) = {} & -2\pi^2\eta(h-h_0)(X)\,F(X) \\ & +2\mathrm{i}\beta\sigma^{-2}\hat{k}_-A_-\int_0^\infty\frac{\exp\left(-2\mathrm{i}\beta\sigma^{-2}\hat{k}_-V\right)}{\sinh\left(V+\frac{1}{2}(X+R_0)\,\beta^{-1}\right)}\,\mathrm{d}V \\ & -2\mathrm{i}\beta\sigma^{-2}\hat{k}_-B_-\int_0^\infty\frac{\exp\left(2\mathrm{i}\beta\sigma^{-2}\hat{k}_-V\right)}{\sinh\left(V+\frac{1}{2}(X+R_0)\,\beta^{-1}\right)}\,\mathrm{d}V \\ & -2\mathrm{i}\beta\sigma^{-2}\hat{k}_+A_+\int_0^\infty\frac{\exp\left(2\mathrm{i}\beta\sigma^{-2}\hat{k}_+V\right)}{\sinh\left(V+\frac{1}{2}(R_1-X)\,\beta^{-1}\right)}\,\mathrm{d}V,\end{aligned} \tag{E 4}$$

and
$$G_2(X) = \tfrac{1}{2}\sigma^{-2} \int_{-R_0}^{R_1} (h-h_0)(V)\, F(V)\, (\coth \tfrac{1}{2}\beta^{-1}(V-X) - \operatorname{sgn}(V-X))\, \mathrm{d}V$$
$$-\tfrac{1}{2}\beta\sigma^{-2} h_- A_- \int_0^\infty \exp(-\mathrm{i}\beta\sigma^{-2}\hat{k}_- V)\, (\coth \tfrac{1}{2}(V+(X+R_0)\,\beta^{-1}) - 1)\, \mathrm{d}V$$
$$-\tfrac{1}{2}\beta\sigma^{-2} h_- B_- \int_0^\infty \exp(\mathrm{i}\beta\sigma^{-2}\hat{k}_- V)\, (\coth \tfrac{1}{2}(V+(X+R_0)\,\beta^{-1}) - 1)\, \mathrm{d}V$$
$$+\tfrac{1}{2}\beta\sigma^{-2} h_+ A_+ \int_0^\infty \exp(\mathrm{i}\beta\sigma^{-2}\hat{k}_+ V)\, (\coth \tfrac{1}{2}(V+(R_1-X)\,\beta^{-1}) - 1)\, \mathrm{d}V$$
$$-\pi^{-2}\beta^{-1} A_- \int_0^\infty \frac{\exp(-2\mathrm{i}\beta\sigma^{-2}\hat{k}_- V)}{\sinh(V+\tfrac{1}{2}(X+R_0)\,\beta^{-1})}\, \mathrm{d}V$$
$$-\pi^{-2}\beta^{-1} B_- \int_0^\infty \frac{\exp(2\mathrm{i}\beta\sigma^{-2}\hat{k}_- V)}{\sinh(V+\tfrac{1}{2}(X+R_0)\,\beta^{-1})}\, \mathrm{d}V$$
$$+\pi^{-2}\beta^{-1} A_+ \int_0^\infty \frac{\exp(2\mathrm{i}\beta\sigma^{-2}\hat{k}_+ V)}{\sinh(V+\tfrac{1}{2}(R_1-X)\,\beta^{-1})}\, \mathrm{d}V. \qquad \text{(E 5)}$$

References

Abramowitz, M. & Stegun, I. A. 1964 *Handbook of mathematical functions with formulas, graphs and mathematical tables.* Washington: National Bureau of Standards.
Bartholomeusz, E. F. 1958 *Proc. Camb. Phil. Soc.* **54**, 106–118.
Carrier, G. F. 1966 *J. Fluid Mech.* **24**, 641–659.
Gradshteyn, I. S. & Ryzhik, I. M. 1965 *Tables of integrals, series and products.* New York: Academic Press.
Kajiura, K. 1963 I.U.G.G. monograph **24**.
Kantorovich, L. V. & Krylov, V. I. 1958 *Approximate methods of higher analysis.* Groningen: Noordhoff.
Keller, J. B. 1958 *J. Fluid Mech.* **4**, 607–614.
Kober, H. 1952 *Dictionary of conformal representations.* New York: Dover.
Kreisel, G. 1949 *Q. appl. Maths.* **7**, 21–44.
Krylov, V. I. 1962 *Approximate calculation of integrals.* New York: Macmillan.
Lighthill, M. J. 1964 *Introduction to Fourier analysis and generalised functions.* Cambridge University Press.
Mahony, J. J. 1967 *Q. appl. Maths.* **25**, 313–316.
Mei, C. C. & Black, J. L. 1969 *J. Fluid Mech.* **38**, 499–511.
Newman, J. N. 1965 *J. Fluid Mech.* **23**, 23–29.
Ogilvie, T. F. 1960 *Inst. Engng Res.*, ser. no. 82, issue no. 14. University of Calif.
Roseau, M. 1952 *Publs scient. tech. Minist. Air*, no. 275.
Ursell, F. 1953 *Proc. Camb. Phil. Soc.* **49**, 685–694.
Wehausen, J. V. & Laitone, E. V. 1960 Surface waves. *Handbuch der Physik* **9**, 446–778.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES